

Reconstruction of colourings without freezing

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October 11, 2016

Abstract

We prove that reconstruction in the k -colouring model occurs strictly below the threshold for freezing for large k .

1 Introduction

The study of broadcast models or spin systems on trees arises naturally in many areas including probability, combinatorics and statistical physics as well as in more applied settings such as computational evolutionary biology and information theory. The so-called reconstruction problem asks when the mutual information between the root and the spins at level ℓ is bounded away from 0 as $\ell \rightarrow \infty$ and thus can be viewed as a type of point to set dependence (see definition in § 1.1). It emerges in numerous settings, for example in biology it determines a phase transition for the information requirements for phylogenetic reconstruction [7].

Here we are most interested in the role the reconstruction threshold plays in the study of random constraints satisfaction problems (rCSPs). It has been shown that in a range of rCSPs such as random colourings of random graphs, the space of solutions undergoes what physicists call a dynamical phase transition in which the space of solutions splits into exponentially many small, isolated clusters [1]. This transition also seems closely related to computational barriers for algorithms for finding solutions. It has been conjectured that the threshold for this transition is exactly the reconstruction threshold and this is known up to first order asymptotic.

Locating the exact reconstruction threshold has only been achieved in a small number of spin systems, the symmetric [14] and near-symmetric binary channels [3] and the three state symmetric channel with large degrees [25]. For the k -colouring model only bounds are known which match in

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the first and second order asymptotic term. On the d -regular tree the model is non-reconstructible whenever [2, 24],

$$d \leq k(\log k + \log \log k + 1 - \log 2 + o_k(1)). \quad (1.1)$$

The best previous bound for reconstruction is when

$$d \geq k(\log k + \log \log k + 1 + o_k(1)) \quad (1.2)$$

by [22, 23]. This uses the following simple algorithm; it reconstructs the root only when it is uniquely determined by the leaves, in which case we say the root is *frozen*. This can be implemented and analysed using a simple recursion and leaves a gap of width just $k \log 2$. It is known that (1.2) is tight for freezing of the root so one natural question to ask is whether reconstruction is possible when the root is not frozen. We answer this in the positive showing that the k -colouring model is still reconstructible for parameters in a small but non-vanishing region of width δk below the freezing threshold.

Interpreted in the setting of random colourings on random graphs this opens a number of tantalising questions. It suggests a range of parameters in which there is clustering of colourings but where the clusters are unfrozen meaning that all vertices can take every possible colour within the cluster. It remains an important question to understand what leads to the computational difficulty in colouring random graphs, the onset of clustering or of freezing. Our result separates these two transitions making this distinction of keen importance.

1.1 Definition and Main Results

The broadcast model on trees is the process where information is sent from the roots downward, along edges acting as noisy channels, to the leaves of the trees. Given a tree $T = (V, E)$, a finite set $[k] = \{1, \dots, k\}$ of k values and a $[k] \times [k]$ probability matrix M as the noisy channel, the broadcast model on tree T is the probability measure on the space of configurations $[k]^V$ defined as follows: The spin σ_ρ at the root ρ is chosen according to the stationary distribution of M , denoted by π . Then for each vertex $v \in T$ with parent u , the spin σ_v is chosen according to the conditional distribution $P(\sigma_v = i \mid \sigma_u = j) = M(i, j)$. In this paper we will focus on the colouring model with alphabet $[k]$ and probability matrix $M(i, j) = \frac{1}{k-1} 1\{i \neq j\}$.

Equivalently, one can also define the colouring model by its Gibbs measure. A proper k -colouring of the graph $G = (V, E)$ is a configuration $\sigma : V \rightarrow [k]$ such that for every edge $e = (u, v) \in E$, $\sigma_u \neq \sigma_v$. The (free) Gibbs measure of random colourings is given by the uniform measure

$$P(\sigma) = \frac{1}{Z} \prod_{e=(u,v) \in E} 1\{\sigma_u \neq \sigma_v\},$$

where Z is the normalizing constant equaling to the number of proper colourings of G .

For technical convenience and also of independent interest, we allow randomness in the underlying trees. For any probability distributions ξ on the set of non-negative integers \mathbb{Z}_+ , we let \mathcal{T}_ξ denote the distribution of Galton-Watson tree with offspring distribution ξ . Two special cases of interest are the d -ary tree \mathcal{T}_d and the Galton-Watson tree $\mathcal{T}_{\text{Pois}(d)}$ with Poisson offspring distribution of average degree d . They are the natural tree models to study with regard to random d -regular graphs and Erdős-Rényi random graphs respectively. The definition of broadcast model can be easily generalized to the (first finite levels of) Galton-Watson trees.

Given a (possibly random) infinite tree, the reconstruction problem asks if the distribution of the state of the root is affected by the configuration on the n 'th level as n goes to infinity. More precisely, let T_n be the first n levels of tree T and L_n be its set of vertices at level n . Write $T_n = T, L_n = \emptyset$ if T has fewer than n levels.

Definition (Reconstruction). Given a family of Galton-Watson trees \mathcal{T}_ξ , we say that the k -colouring model is reconstructible for \mathcal{T}_ξ if there exist $i, j \in [k]$ such that,

$$\limsup_{n \rightarrow \infty} \mathbb{E}_{T \sim \mathcal{T}_\xi} d_{\text{TV}}(P(\sigma_{L_n} = \cdot \mid T, \sigma_\rho = i), P(\sigma_{L_n} = \cdot \mid T, \sigma_\rho = j)) > 0,$$

where d_{TV} is the total-variation distance. Otherwise we say that the model is non-reconstructible.

Non-reconstruction implies that on average the configurations on the distant levels have a vanishing effect on the root. Equivalently, it corresponds to the mutual information between the root and the leaves going to 0 (see e.g. [21] for more equivalent definitions). The freezing threshold is defined as follows:

Definition (Freezing). Given a family of Galton-Watson tree \mathcal{T}_ξ , we say that the k -colouring model is frozen for \mathcal{T}_ξ if

$$\limsup_{n \rightarrow \infty} P_{T \sim \mathcal{T}_\xi}(\sigma_\rho \text{ is uniquely determined by } \sigma_{L_n}) > 0.$$

The exact location of freezing threshold for Poisson tree $\mathcal{T}_{\text{Pois}(d)}$ has been calculated in [19]. Following a similar calculation for \mathcal{T}_d , one can show that for $k \geq k_0$, the k -colouring model is frozen if and only if

$$d > d_k^f := \begin{cases} \inf_{x>0} x \log^{-1} \left(1 - \frac{(1-e^{-x})^k}{k-1} \right) & \mathcal{T}_d \\ \inf_{x>0} \frac{(k-1)x}{(1-e^{-x})^k} & \mathcal{T}_{\text{Pois}(d)} \end{cases} = k(\log k + \log \log k + 1 + o_k(1)).$$

It is easy to see that the k -colouring problem is reconstructible on \mathcal{T}_ξ if it is frozen. Indeed, the freezing threshold gives the best known upper bound for reconstruction threshold with the only exception of $d = 5$ and $k = 14$, in which case reconstruction is proved in [16] using a variational principle. The main result of this paper is the following theorem which implies that the reverse

statement is not true. Throughout we will assume that k exceeds a large enough absolute constant k_0 , where the exact value of k_0 may vary from place to place.

Theorem 1.1. *There exists a constant $\beta^* < 1$ such that for any $k \geq k_0$ the k -colouring model is reconstructible for both \mathcal{T}_d and $\mathcal{T}_{\text{Pois}(d)}$ for d satisfying*

$$d \geq k(\log k + \log \log k + \beta^*). \quad (1.3)$$

For a complete picture, it has been shown [2, 24] that the k -colouring problem is non-reconstructible on d -ary tree with

$$d < k(\log k + \log \log k + 1 - \log 2 + o_k(1)),$$

and the similar result extends to general Galton-Watson trees under mild restrictions [13]. While numerical results of [28] suggest that the actual reconstruction threshold has a constant term roughly in the middle of $1 - \log 2$ and 1, for technical reasons we only show reconstruction for β_* close to the freezing threshold 1. Nonetheless, we believe that our result is of interest because it suggests a distinct phase transition in the solution space evolution of rCSPs, the existence of which was previously unclear. We will address this point in detail in the next section.

1.2 Motivation from Statistical Physics

Random instances of constraint satisfaction problems (rCSPs) have been studied in different areas including theoretical computer science, probability theory, combinatorics and statistical physics. Much of our understanding of the problem over the last two decades comes from the replica/cavity method originally developed by statistical physicist in study of spin glasses, among which perhaps the two most important questions are *when does a rCSP have solution* and *how can we find/sample one*. Significant progresses have been made in the last couple of years towards the first question. Exact satisfiability thresholds have been established for k -NAESAT [10], maximum independent set [8] and k -SAT [9], and the k -colourability threshold has been located within an interval of length $(2 \log 2 - 1)$ [5, 6].

Meanwhile, on the algorithmic side, it has been observed for many models of interest that all polynomial-time algorithms fail to find solutions at densities far below the satisfiability threshold. This algorithmic barrier is believed to be closely related to the phase transitions in the geometry of the set of solutions. Here we briefly review the heuristic phase diagram developed by statistical physicists [15, 28], as we fix k and increase the average connectivity d . The set of solutions start out as a well-connected component containing all but exponentially small fraction of solutions. At the *clustering threshold* d_{clust} , the solution space splits into an exponential number of “clusters” where clusters are well-connected inside but well-separated from each other, and no single cluster contains more than an exponentially small fraction of all solutions. Then at the possibly higher

value of d , namely the *rigidity threshold* d_{rigid} , typical clusters become “frozen”, i.e. a linear fraction of variables take the same value in all solutions of that cluster. Finally at much larger values of d come the condensation threshold and satisfiability threshold, which we will not go into details here.

These predictions have been partially verified in many cases. Apart from the results on satisfiability threshold mentioned before, Molloy [19] proved that the rigidity phase transition coincide with the freezing threshold on trees, in the case of k -colouring. And in the prominent paper [1], the authors proved that the solution space does split into exponentially many frozen clusters for k -colourings models and constraint densities $(1 + o_k(1))k \log k \leq d \leq (2 - o_k(1))k \log k$.

Among the different phase transitions mentioned above, it has been conjectured that the clustering threshold and the rigidity threshold are the two factors resulting the onset of hard random-CSP instances. However different opinions exist on which one is more responsible [18, 28, 29], if any of them [4]. And much is unclear about how they affect the performance of algorithms directly. One difficulty lies in the fact that the two thresholds are extremely close to each other. According to the physics prediction, [28], both thresholds happen at $k(\log k + \log \log k + \alpha + o(1))$ for different values of α and no evidence shows even at a heuristic level that such gap is indeed non-vanishing. In fact, it has been widely believed that the clustering phase transition, marking the onset of long range correlation, coincides with the reconstruction threshold on trees [20]. If that is the case, then previous results in the reconstruction problems [24] imply that the gap between the two thresholds can at most be $k(\log 2 + o(1))$ (compared to the leading term of $k \log k$).

We hope that the result of this paper can contribute to the understanding of colourings on random graphs in two directions. First, we show for the first time that the gap between reconstruction threshold and freezing threshold on trees is linear in k . This combined with the conjecture that reconstruction coincides with clustering strongly suggests a distinct phase where the solution space are clustered but non-frozen. It will be of great interest to analyze algorithms in this region. Secondly, the distributional recursion involved in the reconstruction problem (known as the averaged 1RSB equation in physics jargon [17]) is closely related to the BP recursion, thus in bounding the fixed point of the reconstruction recursion, we hope to provide additional information on the fixed point of the BP recursion, and in turn improve the understanding of the structure of the clusters.

We conclude this section by noting the implication of our results for sampling algorithms, as non-reconstruction is closely related to the efficiency of MCMC. Typically, local algorithms are efficient only when there is no long-range correlation. Recently, it was shown that Glauber dynamics of k -colouring model on d -ary trees has $O(n \log n)$ mixing time in the entire non-reconstruction regime [26]. Much less is known on random graphs (Erdős-Rényi, random d -regular graph, etc.). The best bounds for efficient algorithms so far are $k \geq 5.5d$ using the Glauber dynamics are [12] and $k \geq 3d$ using non-MCMC methods [27], both of which are still below the uniqueness threshold.

1.3 Outline of the proof

The proof of Theorem 1.1 essentially follows from a detailed analysis of the tree recursion. We begin by specifying the distribution of the reconstruction probability $P(\sigma_\rho = \cdot \mid \sigma_{L_n})$ on n -level trees as a function of the distribution on $(n-1)$ -level trees $P(\sigma_\rho = \cdot \mid \sigma_{L_{n-1}})$. This defines a distributional recursion on the set of probability measures on the k dimensional simplex Δ^k . For the purpose of proving reconstruction, it is enough to show that the recursion has a non-trivial fixed point, which is done in two steps: First we show that there exists a non-trivial measure μ on Δ^k such that after one step of the recursion the new measure stochastically dominates the original one. This step is done in Section 3. Given the result of stochastic dominance, we provide a randomized algorithm such that the distribution of the reconstruction probability equals μ on trees of any depth, which is done in Section 2.

2 Reconstruction algorithm

We begin by introducing the notations we will be using throughout the proof. In general, we will use $U, V \dots$ for random variables and μ, ν for measures. To avoid complicated subscripts, we will use both U and μ_U for the distribution of U and use f_U for its density (using delta functions for atoms). For any function φ , we write $\varphi \circ \mu$ for the distribution of $\varphi(X)$, where X is a random sample of μ , denoted as $X \sim \mu$. We will use $B \oplus C$ to denote the (measure of) the sum of two independent copies of B and C , and $a \otimes B$ to denote the sum of a i.i.d. copies of B . One should distinguish these two operators with $+$ and \cdot , the usual addition and scalar multiplication of measures. By definition, we have

$$\mu_{B \oplus C} = \mu_B * \mu_C, \quad \mu_{a \otimes B} = \underbrace{\mu_B * \mu_B * \dots * \mu_B}_{a \text{ times}}.$$

For any space Ω , we will use $\mathcal{M}(\Omega)$ to denote the space of probability measures on Ω . A substantial portion of our proof will be comparing different measures. For that sake, we define the following partial order on $\mathcal{M}(\overline{\mathbb{R}})$, where $\overline{\mathbb{R}} \equiv \mathbb{R} \cup \{-\infty, \infty\}$ is the extended real numbers.

Definition 2.1 (Stochastic dominance). For $\mu, \nu \in \mathcal{M}(\overline{\mathbb{R}})$, we say that ν *stochastically dominates* μ , denoted by $\mu \prec \nu$, if for any $x \in \overline{\mathbb{R}}$, $\mu([-\infty, x]) \geq \nu([-\infty, x])$. Moreover, for any $\epsilon > 0$, we say that ν *stochastically dominates μ by ϵ* , denoted by $\mu \prec_\epsilon \nu$, if for any $x \in \overline{\mathbb{R}}$, we have either $\mu([-\infty, x]) = 1$ or $\nu([-\infty, x]) = 0$ or $\mu([-\infty, x]) - \epsilon \geq \nu([-\infty, x])$.

The following proposition gives two sufficient conditions of stochastic dominance that will be used throughout the proof. The proof of proposition should be trivial.

Proposition 2.2. *Let X, Y be two arbitrary independent random variables*

1. If μ_X, μ_Y are absolutely continuous and $f_X(y) \leq f_Y(y)$ for all y satisfying $P(Y \geq y) > 0$, then $X \succ Y$.
2. If X stochastically dominates Y by ϵ , then for any random variable X' such that $P(X \neq X') \leq \epsilon$ and $\{x'; P(X' < x') = 0\} \subseteq \{y; P(Y < y) = 0\}$, X' also stochastically dominates Y .

2.1 k -colouring model and the tree recursion

In this section we give the distributional recursion involved in the reconstruction problem. Let $[k] = \{1, \dots, k\}$ denote the set of k -colours and $T = (V, E) \sim \mathcal{T}_\xi$ be an instance of the Galton-Watson tree of offspring distribution ξ with root ρ . For each $n \geq 1$, let $T_n = (V_n, E_n)$ denote the restriction of T to its first n levels and let L_n be the leaves of T_n . For each n , the k -colouring model restricted on T_n is the uniform measure on the set of proper colourings

$$\Omega_n := \{\sigma \in [k]^{V_n} : \sigma_u \neq \sigma_v, \text{ for all } e = (u, v) \in E_n\}.$$

And we will use $\Omega(L_n)$ to denote the set of possible configurations on L_n .

For any $\eta \in \Omega(L_n)$ and $l \in [k]$, let f_n be the (deterministic) function defined as follows:

$$f_n(l, \eta; T) := P(\sigma_\rho = l | T_n, \sigma_{L_n} = \eta).$$

Given tree T_n and the observed configuration $\eta \in \Omega(L_n)$, the maximum likelihood estimator of σ_ρ is the colour l that achieves the maximum of $f_n(l, \eta; T)$, and this estimation is correct with probability $\max_l f_n(l, \eta; T)$. Let d_ρ be the degree of the root ρ of T , and u_1, \dots, u_{d_ρ} be the d_ρ offspring of the root ρ . For each $1 \leq i \leq d_\rho$, let T_i be the subtree rooted at u_i and $L_i^n = L_n \cap T_i$ be the subset of L_n restricted to T_i . Given the colour of u_i , the configuration on T_i is independent of the configuration on $T \setminus T_i$. A standard recursive calculation gives that, for each $\eta \in \Omega(L_n)$ and $l \in [k]$,

$$f_{n+1}(l, \eta; T) = \frac{\prod_{i=1}^{d_\rho} (1 - f_n(l, \eta_i, T_i))}{\sum_{m=1}^k \prod_{i=1}^{d_\rho} (1 - f_n(m, \eta_i, T_i))}. \quad (2.1)$$

To study one step of the recursion from a vertex, one first samples the number of offspring from ξ then decides the colour of each offspring accordingly. Let $\Xi^l = \Xi^l(n; \xi)$ denote the distribution of (T_n, σ_{L_n}) given $\sigma_\rho = l$ and let (T_n, η^l) be a sample from Ξ^l . Then the vector of posterior probability $\vec{X}_n := (f_n(1, \eta^1; T), \dots, f_n(k, \eta^1; T))$ is a random vector in the k -dimensional simplex $\Delta^k := \{(x_1, \dots, x_k) : x_i \geq 0, \sum_{i=1}^k x_i = 1\}$. Let (T_i, η_i^l) be the restriction of (T_n, η^l) onto T_i . By the symmetry between branches of Galton-Watson trees and the symmetry between colours, we have that

$$(f_n(m, \eta^l; T))_{m=1}^k \stackrel{d.}{=} (X_n^{(m-l+1)})_{m=1}^k,$$

where we use the notation $x^{(l)}$ to denote the l -th entry of vector \vec{x} , modulo k when necessary. Furthermore, conditioned on the value of $\vec{X}_n^{(1)}$, $(\vec{X}_n^{(2)}, \dots, \vec{X}_n^{(k)})$ are exchangeable. In particular $\vec{X}_n^{(l)} \stackrel{d}{=} \vec{X}_n^{(2)}$ for all $l \neq 1$.

The distribution of \vec{X}_n can be solved recursively using the following Δ^k -valued function Γ that takes an indefinite number of variables: Let

$$\Gamma^{(m)}(\vec{x}_{i,l}, l = 1, \dots, k, i = 1, \dots, b_l) := \frac{\prod_{l=2}^k \prod_{i=1}^{b_l} (1 - \vec{x}_{i,l}^{(m-l+1)})}{\sum_{l'=1}^k \prod_{l=2}^k \prod_{i=1}^{b_l} (1 - \vec{x}_{i,l}^{(l'-l+1)})}, \quad \forall m \in [k], \quad (2.2)$$

where we adopt the convention of $\prod_{i \in \emptyset} a_i = 1$. Here b_l represent the number of u_i 's with colour l . Given d_ρ and $\sigma_\rho = 1$, the joint distribution of (b_2, \dots, b_k) follows the multinomial distribution with sum d_ρ and probability $(\frac{1}{k-1}, \dots, \frac{1}{k-1})$ and $b_1 = 0$. Let $D_\rho, (B_1, \dots, B_k)$ be an i.i.d. copy of $d_\rho, (b_1, \dots, b_k)$ and $\vec{X}_{i,l}$ be i.i.d. samples of \vec{X}_n , (2.1) implies that

$$\vec{X}_{n+1} \stackrel{d}{=} \left(\frac{\prod_{l=2}^k \prod_{i=1}^{B_l} (1 - \vec{X}_{i,l}^{(m-l+1)})}{\sum_{m'=1}^k \prod_{l=2}^k \prod_{i=1}^{B_l} (1 - \vec{X}_{i,l}^{(m'-l+1)})} \right)_{m=1}^k = \Gamma(\vec{X}_{i,l}, l = 1, \dots, k, i = 1, \dots, B_l). \quad (2.3)$$

Let $\tilde{\Xi}$ be the distribution of (T_n, σ_{L_n}) without conditioning on the value of σ_ρ and define the unconditional posterior probability $\tilde{X}_n := (f_n(1, \tilde{\eta}; T), \dots, f_n(k, \tilde{\eta}, T))$ similarly, where $\tilde{\eta}$ is sampled from $\tilde{\Xi}$. The distribution of \vec{X}_n and \tilde{X}_n satisfies that at each point $x \in \Delta^k$,

$$\begin{aligned} P(\vec{X}_n \in dx) &= kP\left(\sigma_\rho = 1, (P(\tau_\rho = j \mid T_n, \tau_{L_n} = \sigma_{L_n}))_{j=1}^k \in dx\right) \\ &= kP(\tilde{X}_n \in dx)P(\sigma_\rho = 1 \mid (P(\tau_\rho = j \mid T_n, \tau_{L_n} = \sigma_{L_n}))_{j=1}^k \in dx) \\ &= kx^{(1)}P(\tilde{X}_n \in dx). \end{aligned} \quad (2.4)$$

Equation (2.3) and (2.4) are all we need to describe the distributional recursion. To be more concrete, we introduce some further notations. Let $\mathcal{M}_s(\Delta^k) \subset \mathcal{M}(\Delta^k)$ be the subset of measures in $\mathcal{M}(\Delta^k)$ that are invariant under permutations of the coordinates. With some abuse of notation, we will also use Γ for the transformation it induces on $\mathcal{M}(\Delta^k)$, i.e. for any $\nu \in \mathcal{M}(\Delta^k)$, we define $\Gamma\nu$ as the distribution of $\Gamma(\vec{X}_{i,l}, l = 1, \dots, k, i = 1, \dots, B_l)$ where $\vec{X}_{i,l}$ are i.i.d. copies with distribution ν and B_l are defined as before. For each $\nu \in \mathcal{M}_s(\Delta^k)$, let $\Pi_l\nu$ be defined as $(\Pi_l\nu)(dx) := kx^{(l)}\nu(dx)$ and define

$$\Gamma_s\nu := \frac{1}{k} \sum_{l=1}^k (\Gamma \circ \Pi_l)\nu. \quad (2.5)$$

Under these notations, if $\tilde{X}_n \sim \nu$, then $\vec{X}_n \sim \Pi_1\nu$, $\vec{X}_{n+1} \sim \Gamma \circ \Pi_1\nu$ and $\vec{X}_{n+1} \sim \Gamma_s\nu$.

It is easy to check that $\delta_{(\frac{1}{k}, \dots, \frac{1}{k})}$ is a trivial fixed point of Γ_s , which corresponds to no information about the root. To show reconstruction, it is enough to prove for $\vec{X}_0 \sim \mu_0 := \frac{1}{k}[\delta_{(1,0,\dots,0)} + \dots +$

$\delta_{(0,\dots,0,1)}]$ that $\Gamma_s^n \mu_0$ is weakly bounded away from $\delta_{(\frac{1}{k},\dots,\frac{1}{k})}$. One of the main difficulties for analyzing graph colourings is that the dimension of the recursion grows linearly in k . Luckily, as it will become clear in the proof, it is sufficient to consider only the largest coordinate of \tilde{X}_n . All the other entries are w.h.p. negligible as $k \rightarrow \infty$. Since we are not aiming at the tightest possible bound, we shall discard this extra information reducing the recursion to \mathbb{R} .

Define $\lambda(\vec{x}) = (\lambda^{(0)}, \lambda^{(1)})(\vec{x}) := (\|\vec{x}\|_\infty, \arg \max \vec{x})$ and $\Lambda : \Delta^k \rightarrow \Delta^k$ to be

$$\Lambda^{(m)}(\vec{x}) = \begin{cases} \|\vec{x}\|_\infty & m = \arg \max \|\vec{x}\|_\infty \\ \frac{1-\|\vec{x}\|_\infty}{k-1} & \text{otherwise} \end{cases}. \quad (2.6)$$

We are mostly interested in the transformation λ and Λ induces on spaces of probability measures. With some abuse of notation, we allow extra randomness to be used to break ties in the $\arg \max$ of λ and Λ independently and uniformly randomly. For example if $X = (\frac{1}{2}, \frac{1}{2}, 0, \dots, 0)$ with probability 1, then $\lambda(X)$ equals $(\frac{1}{2}, 1)$ or $(\frac{1}{2}, 2)$ with probability $\frac{1}{2}$. Let $\Lambda^k = \Lambda(\Delta^k) \subset \Delta^k$ be the “star-shaped” image of Λ , $\lambda(\vec{x})$ gives a bijection between $\Lambda^k \setminus (\frac{1}{k}, \frac{1}{k}, \dots, \frac{1}{k})$ and $(\frac{1}{k}, 1] \times [k]$. Hence there is a bijection between $\mathcal{M}([\frac{1}{k}, 1])$ and $\mathcal{M}_s(\Lambda^k) := \mathcal{M}_s(\Delta^k) \cap \mathcal{M}(\Lambda^k)$ given by:

$$\begin{aligned} \lambda^{(0)} : \mathcal{M}_s(\Lambda^k) &\rightarrow \mathcal{M}([\frac{1}{k}, 1]), \quad \mu \rightarrow \lambda^{(0)} \circ \mu = \|\mu\|_\infty; \\ \lambda^{-1} : \mathcal{M}([\frac{1}{k}, 1]) &\rightarrow \mathcal{M}_s(\Lambda^k), \quad \mu \rightarrow \lambda^{-1} \circ \left(\mu \otimes \frac{1}{k}(\delta_1 + \dots + \delta_k) \right). \end{aligned}$$

Thus $\Lambda \circ \Gamma_s$ induces a transformation on $\mathcal{M}_s(\Lambda^k)$ and $\lambda^{(0)} \circ \Lambda \circ \Gamma_s \circ \lambda^{-1} = \|\Gamma_s \circ \lambda^{-1}\|_\infty$ induces a transformation on $\mathcal{M}([\frac{1}{k}, 1])$. With another abuse of notation, we will use the same notation for both $\mu \in \mathcal{M}([\frac{1}{k}, 1])$ and its unique correspondence in $\mathcal{M}_s(\Lambda_k)$ and use $\Lambda \circ \Gamma_s$ for both transformations. Also for $\mu, \nu \in \mathcal{M}_s(\Lambda_k)$, we say $\mu \prec \nu$ iff $\mu \prec \nu$ as elements of $\mathcal{M}([\frac{1}{k}, 1])$.

The main technical result of this paper is the following theorem, which will be proved in Section 3.

Theorem 2.3. *There exist $\beta^0 < 1, c > 0$ such that for any $k > k_0$, $d \geq k(\log k + \log \log k + \beta^0)$, and $T \sim \mathcal{T}_{\text{Pois}(d)}$, one can construct $\mu_k \in \mathcal{M}([\frac{1}{k}, 1])$ such that $(\Lambda \circ \Gamma_s)\mu_k$ stochastically dominates μ_k by $c/\log k$.*

Using the fact that $\|\Lambda(\vec{x})\|_\infty = \|\vec{x}\|_\infty$, Theorem 2.3 is equivalent to the statement that $\|\Gamma_s \mu_k\|_\infty$ stochastically dominates μ_k by $c/\log k$. It follows that if at some level we can reconstruct the root with success probability $\|\tilde{X}_n\|_\infty$ for some $\tilde{X}_n \sim \mu_k \in \mathcal{M}_s(\Lambda^k)$, then in the level above we can do strictly better with success probability $\|\tilde{X}_{n+1}\|_\infty \succ \|\tilde{X}_n\|_\infty$. However this does not directly imply reconstruction due to two reasons. First, the proof of Theorem 2.3 depends heavily on the low-dimensional structure of $\mu_k \in \mathcal{M}_s(\Lambda^k)$, but in general after one step $\Gamma_s \mu_k$ no longer belongs to $\mathcal{M}_s(\Lambda_k)$. Secondly, due to the non-linearity of $\Lambda \circ \Gamma_s$, it is not clear whether $(\Lambda \circ \Gamma_s)\mu_k \succ \mu_k$ would imply $(\Lambda \circ \Gamma_s)^2 \mu_k \succ (\Lambda \circ \Gamma_s)\mu_k$. We address both problems in next subsection by intentionally

manipulating the observed configuration and thus manually maintaining a nontrivial fixed point for the “manipulated recursion”.

2.2 Manipulating the tree recursions

In this section we provide a reconstruction algorithm such that its estimator of σ_ρ satisfies a modified recursion with the fixed point μ_k defined in Theorem 2.3. Let \mathbf{S}_k be the symmetric group of degree k . For any $\pi \in \mathbf{S}_k$, $\eta \in \Omega(L_n)$ and $X \in \Delta^k$, define $\pi \circ \eta \in \Omega(L_n)$ to be the configuration specified by $(\pi \circ \eta)_v = \pi(\eta_v)$ and $\pi \circ X \in \Delta^k$ to be the vector with $(\pi \circ X)^{(l)} = X^{(\pi(l))}$. We first illustrate the main idea with an example:

Suppose that two people, Alice and Bob, are trying to reconstruct σ_ρ , the colour of the root, from σ_{L_n} . Observing T and $\sigma_{L_n} = \eta \in \Omega(L_n)$, Bob knows that root ρ has colour l with probability $f_n(l, \eta; T)$. Then Alice tells Bob that the η he observed was not the actual σ_{L_n} , but the σ_{L_n} after a randomly selected permutation π . Namely, $\eta = \pi \circ \sigma_{L_n}$ where π is sampled from some distribution $\nu \in \mathcal{M}(\mathbf{S}_k)$. Let $F(\eta) := (f_n(\ell, \eta; T))_{\ell=1}^k \in \Delta^k$ be the original estimator of the root with T omitted for brevity. Bob’s estimation of σ_ρ after Alice’s permutation becomes

$$F(\eta; \nu) := \left(P_{\pi \sim \nu}(\sigma_\rho = l \mid \pi \circ \sigma_{L_n} = \eta) \right)_{l=1}^k = \sum_{\pi \in \mathbf{S}_k} \nu(\pi) F(\pi^{-1} \circ \eta) = \sum_{\pi \in \mathbf{S}_k} \nu(\pi) (\pi \circ F)(\eta).$$

Thus if Alice chooses the distribution ν carefully, she can manipulate Bob’s estimation to any vector in the convex hull of $\{(\pi \circ F)(\eta) : \pi \in \mathbf{S}_k\}$. And that’s essentially what we will do in this section. In particular, we consider the following two families of $\nu \in \mathcal{M}(\mathbf{S}_k)$:

1. For each $l \in [k]$, let $\nu_1(l)$ be the uniform distribution on $\mathbf{S}_{[k] \setminus l} := \{\pi \in \mathbf{S}_k : \pi_l = l\}$. For any $\eta \in \Omega(L_n)$ and $m \in [k]$,

$$F^{(m)}(\eta; \nu_1(l)) = \begin{cases} f_n(m, \eta) & m = l \\ \frac{1}{k-1} \sum_{m \neq l} f_n(m, \eta) & m \neq l \end{cases} = \begin{cases} f_n(m, \eta) & m = l \\ \frac{1}{k-1} (1 - f_n(m, \eta)) & m \neq l \end{cases}. \quad (2.7)$$

2. For each $p \in [0, 1]$, let $\nu_2(p) := p\nu_{\text{unif}} + (1-p)\delta_{\text{id}}$ where ν_{unif} is the uniform distribution on \mathbf{S}_k and δ_{id} is the point mass at the identity permutation id . For any $\eta \in \Omega(L_n)$,

$$F(\eta; \nu_2(p)) = (1-p)F(\eta) + \frac{p}{k!} \sum_{\pi \in \mathbf{S}_k} (\pi \circ F)(\eta) = (1-p)F(\eta) + p \cdot \left(\frac{1}{k}, \dots, \frac{1}{k} \right). \quad (2.8)$$

In the proof, we will use $\nu_1(l)$ to simulate the transformation Λ defined in (2.6) and $\nu_2(p)$ to reduce the distribution $(\Lambda \circ \Gamma_s)\mu_k$ to μ_k . For the later purpose, we show the following lemma.

Lemma 2.4. For any $\mu_1, \mu_2 \in \mathcal{M}([\frac{1}{k}, 1])$ such that $\mu_1 \succ \mu_2$, there exist function $q : [\frac{1}{k}, 1] \times [0, 1] \rightarrow [\frac{1}{k}, 1]$, such that $q(y, u) \leq y$ for all $y \in [\frac{1}{k}, 1], u \in [0, 1]$ and for any independent random variables $Y \sim \mu_1$ and $U \sim \text{Unif}[0, 1]$, $q(Y, U) \sim \mu_2$. We say that such function q reduces μ_1 to μ_2 .

Proof. Let G_1, G_2 be the c.d.f. of μ_1, μ_2 , and $G_1(x-0)$ be the left limit of G_1 at x . For $y \geq \frac{1}{k}$, define

$$q(y, u) := \inf \left\{ x \geq \frac{1}{k} : G_2(x) \geq G_1(y-0) + u(G_1(y) - G_1(y-0)) \right\}.$$

Note that $\mu_1 \succ \mu_2$ implies that $G_2(y) \geq G_1(y)$ for all $y \geq \frac{1}{k}$. Hence $q(y, u) \in [\frac{1}{k}, y]$. Let $y_x = \sup\{y : G_1(y-0) \leq G_2(x)\}$. A direct calculation shows that for $x \geq \frac{1}{k}$,

$$\begin{aligned} P(q(Y, U) \leq x) &= P(G_2(x) \geq G_1(Y-0) + U(G_1(Y) - G_1(Y-0))) \\ &= G_1(y_x-0) + (G_1(y_x) - G_1(y_x-0)) \frac{G_2(x) - G_1(y_x-0)}{G_1(y_x) - G_1(y_x-0)} = G_2(x). \end{aligned}$$

□

Recalling the 1-to-1 correspondence between $\mathcal{M}(\Lambda^k)$ and $\mathcal{M}([\frac{1}{k}, 1])$, we define q_0 to be the function that reduces $\mu_0 = \frac{1}{k}(\delta_{(1,0,\dots,0)} + \dots + \delta_{(0,\dots,0,1)})$ to μ_k and q_\star to be the function that reduces $(\Lambda \circ \Gamma_s)\mu_k$ to μ_k , where the later one exists because $(\Lambda \circ \Gamma_s)\mu_k \succ \mu_k$. We further define for each $\bullet \in \{0, \star\}$ that

$$\tilde{q}_\bullet(y, u) := \frac{ky - q_\bullet(y, u)}{ky - 1} \in [0, 1] \quad \text{such that} \quad (1 - \tilde{q}_\bullet(y, u)) \cdot y + \tilde{q}_\bullet(y, u) \cdot \frac{1}{k} = q_\bullet(y, u). \quad (2.9)$$

Let us introduce further notations necessary for the algorithm: Let $\mathbf{U} := (U_v)_{v \in T}$ be an array of independent $\text{Unif}[0, 1]$ random variables indexed by the vertices of T and let $\mathbf{U}_v := (U_w)_{w \in T_v}$ be the sub-array indexed over T_v , the subtree rooted at v . For each $v \in T$ and $w \in T_v$, we will encode Alice's action on T_v and Bob's information at w after Alice's actions on T_v as

$$\mathbf{a}_v := (p_v, l_v, \pi_v) \in [0, 1] \times [k] \times \mathbf{S}_k \quad \text{and} \quad \mathbf{b}_{w,v} := (p_{w,v}, \eta_{w,v}) \in [0, 1] \times [k].$$

Let \mathbf{A}_v and \mathbf{B}_v be arrays of \mathbf{a}_w and $\mathbf{b}_{w,v}$ indexed over $w \in T_v$ respectively. Letting L_1^v denote the set of offspring of v , we define $\mathbf{B}_{L_1^v} := (\mathbf{b}_{w,u})_{u \in L_1^v, w \in T_u}$ as the concatenation of $(\mathbf{B}_u)_{u \in L_1^v}$ for each $v \notin L_n$ and define $\mathbf{B}_{L_1^v} := (\sigma_v)$ otherwise. With the meaning of \mathbf{a}_v and $\mathbf{b}_{w,v}$ to be given in a moment, we formally define

$$\mathbf{P}_v^\circ := \mathbf{P}_v^\circ(\mathbf{B}_{L_1^v}) = \begin{cases} (P(\sigma_v = l \mid \sigma_v))_{l=1}^k & v \in L_n, \\ (P(\sigma_v = l \mid \mathbf{B}_{L_1^v}))_{l=1}^k & v \notin L_n. \end{cases}, \quad \mathbf{P}_v := \mathbf{P}_v(\mathbf{B}_v) = (P(\sigma_v = l \mid \mathbf{B}_v))_{l=1}^k,$$

as Bob's belief on σ_v before and after Alice's actions on T_v (if he is given $\mathbf{B}_{L_1^v}$ or \mathbf{B}_v respectively).

We now define the actions of Alice, namely what $\mathbf{a}_v, \mathbf{b}_v$ means and how she recursively constructs them from the leaves up to the root as a function of $T_v, \sigma_{T_v \cap L_n}$ and \mathbf{U}_v :

1. For each leaf vertex $v \in L_n$, $T_v = \{v\}$. Bob's belief before Alice's action is simply

$$\mathbf{P}_v^\circ = (P(\sigma_v = l \mid \sigma_v))_{l=1}^k = (\mathbf{1}\{\sigma_v = l\})_{l=1}^k.$$

Alice then sets $l_v = \sigma_v$, $p_v = \tilde{q}_0(1, U_v)$ and $\pi_v = \pi_v^2 \circ \pi_v^1$, where π_v^1 is a sample of $\nu_1(l_v)$ and π_v^2 is an independent sample of $\nu_2(p_v)$. Finally, she permute σ_v by π_v (which has the same effect as using π_v^2) and prepares Bob's share of information as $\mathbf{B}_v = (\mathbf{b}_{v,v})$, where

$$\mathbf{b}_{v,v} = (q_{v,v}, \eta_{v,v}) = (p_v, \pi_v^2(l_v)) = (p_v, \pi_v^2(\sigma_v)).$$

2. Suppose that for each $w \in L_{m+1}$, Alice has recorded her actions on T_w as \mathbf{A}_w and prepared the information for Bob as \mathbf{B}_w , where \mathbf{A}_w is a function of $(T_w, \sigma_{T_w \cap L_n}, \mathbf{U}_w)$ and \mathbf{B}_w is a function of \mathbf{A}_w . We now describe Alice's actions on T_v , namely how she constructs \mathbf{A}_v and \mathbf{B}_v for each $v \in L_m$ as a function of $(\mathbf{B}_u)_{u \in L_1^v}$ and U_v .

- (a) First, for each $u \in L_1^v$, Alice calculates \mathbf{P}_u , namely Bob's belief of σ_u given information \mathbf{B}_u . Given $(\mathbf{P}_u)_{u \in L_1^v}$, Alice calculates Bob's belief of σ_v before her actions on T_v . Following a similar recursion of (2.1),

$$\mathbf{P}_v^\circ = \left(\frac{\prod_{u \in L_1^v} (1 - \mathbf{P}_u^{(l)})}{\sum_{m=1}^k \prod_{u \in L_1^v} (1 - \mathbf{P}_u^{(m)})} \right)_{l=1}^k.$$

- (b) Let $U_v^i, i = 1, 2, 3$ be three independent $\text{Unif}[0, 1]$ random variables constructed from U_v . Let $l_v = l_v(\mathbf{P}_v^\circ, U_v^1)$ be uniformly picked from $\{l : (\mathbf{P}_v^\circ)^{(l)} = \|\mathbf{P}_v^\circ\|_\infty\}$, the set of largest coordinates of \mathbf{P}_v° , using the randomness of U_v^1 and let $p_v = \tilde{q}_\star(\|\mathbf{P}_v^\circ\|_\infty, U_v^2)$. Alice then uses the randomness U_v^3 to sample π_v^1 from $\nu_1(l_v)$ and π_v^2 from $\nu_2(p_v)$ independently and sets $\pi_v = \pi_v^2 \circ \pi_v^1$. This gives $\mathbf{a}_v = (p_v, l_v, \pi_v)$ and completes the construction of \mathbf{A}_v .
- (c) Finally, Alice "permutes" Bob's current observation of $T_v \cap L_n$ and all the previous information she prepares for Bob by π_v . This, in the language of \mathbf{A}_v and \mathbf{B}_v , corresponds to setting $q_{v,v} = p_v$, $\eta_{v,v} = \pi_v^2(l_v)$ and setting for each $w \in T_v \setminus \{v\}$ that $q_{w,v} = p_w$ and

$$\eta_{w,v} = \pi_v(\eta_{w,w_1}) = \pi_{w_0}(\pi_{w_1}(\cdots \pi_{w_{r-1}}(\pi_w^2(l_w)) \cdots) \cdots),$$

where $w_0 = v, w_1 \in L_1^v, \dots, w_{r-1}, w_r = w$ is the unique path connecting v to w . This completes the definition of $\mathbf{B}_v = (\mathbf{b}_{w,v})_{w \in T_v}$.

3. As a final step, Alice tells Bob the array \mathbf{B}_ρ as partial information of her actions, which in particular includes Bob's final observation as $(\eta_{v,\rho})_{v \in L_n}$. We emphasize that \mathbf{B}_ρ is the only piece of information given to Bob. All the intermediate \mathbf{B}_v 's exist only in Alice's deduction and remain unknown to Bob.

The main result of the section is the following Theorem.

Theorem 2.5. *For any $n \geq 1$, let T be a n -level tree sampled from $\mathcal{T}_{\text{Pois}}$ and σ_{L_n} be generated by the colouring model on T . Let \mathbf{U} be a T -indexed array of independent $\text{Unif}[0, 1]$ random variables. If Alice performs her actions as described above, then Bob's final belief of σ_ρ after all Alice's actions, represented as*

$$\mathbf{P}_\rho = \mathbf{P}_\rho(\mathbf{B}_\rho) = (P(\sigma_\rho = l \mid \mathbf{B}_\rho))_{l=1}^k \in \Delta^k,$$

follows the distribution of μ_k .

Proof. For each permutation $\pi \in \mathbf{S}_k$ and T -indexed array $\mathbf{B} = (\mathbf{b}_v)_{v \in T} \in ([0, 1] \times [k])^T$, let $\pi \circ \mathbf{b}_v := (p_v, \pi(\eta_v))$ and $\pi \circ \mathbf{B} := (\pi \circ \mathbf{b}_v)_{v \in T}$. We induct on the number of levels in tree T to prove the claim of Theorem 2.5 together with the result that

$$\mathbf{P}_\rho(\pi \circ \mathbf{B}_\rho) = \left(P(\sigma_\rho = l \mid \pi \circ \mathbf{B}_\rho) \right)_{l=1}^k = \pi^{-1} \circ \mathbf{P}_\rho(\mathbf{B}_\rho). \quad (2.10)$$

For $n = 0$, $T = \{\rho\}$ is the singleton tree and $\mathbf{P}_\rho^\circ = (\mathbf{1}\{\sigma_\rho = l\})_{l=1}^k$, Bob's belief before Alice's action, follows distribution μ_0 . Given $\mathbf{b}_{\rho,\rho} = (p_\rho, \eta_{\rho,\rho})$, Bob's posterior estimation of σ_ρ satisfies

$$P(\sigma_\rho = \tilde{\pi}^{-1}(\eta_{\rho,\rho}) \mid \mathbf{b}_{\rho,\rho}) = \nu_2(p_\rho)(\tilde{\pi}), \quad \forall \tilde{\pi} \in \mathbf{S}_k.$$

Therefore, applying (2.8), Bob's belief of σ_ρ after Alice's action at ρ becomes

$$\mathbf{P}_\rho = \left(P(\sigma_\rho = l \mid \pi_\rho(\sigma_\rho) = \eta_{\rho,\rho}) \right)_{l=1}^k = (1 - p_\rho) \mathbf{P}_\rho^\circ + p_\rho \cdot \left(\frac{1}{k}, \dots, \frac{1}{k} \right).$$

Observe that by definition $p_\rho = \tilde{q}_0(1, U_\rho) = \tilde{q}_0(\|\mathbf{P}_\rho^\circ\|_\infty, U_\rho)$. Lemma 2.4 and (2.9) then imply that \mathbf{P}_ρ follows the distribution of μ_k . It is not hard to check that (2.10) also holds.

Suppose we have proved Theorem 2.5 and (2.10) for trees no greater than $n - 1$ levels, we now proceed to trees of n levels. By the induction hypothesis, for each $u \in L_1$, $\mathbf{P}_u = \mathbf{P}_u(\mathbf{B}_u)$, Bob's belief of σ_u after Alice's actions on T_u , follows the distribution μ_k . Following a similar calculation of (2.4), we can show that conditioning on $\sigma_\rho = l$ but not T and $\sigma_{T \setminus \{\rho\}}$, $(\mathbf{P}_u)_{u \in L_1}$ has the same joint distribution as $\text{Pois}(d)$ independent samples of $\Pi_l \mu_k$. Therefore

$$\mathbf{P}_\rho^\circ = \left(\frac{\prod_{u \in L_1} (1 - \mathbf{P}_u^{(l)})}{\sum_{m=1}^k \prod_{u \in L_1} (1 - \mathbf{P}_u^{(m)})} \right)_{l=1}^k \sim \Gamma_s \mu_k.$$

Now we turn to $P_\rho = P_\rho(B_\rho)$. For each $u \in L_1$, let $B_{\rho,u} := (b_{w,\rho})_{w \in T_u}$, $B_{\rho,L_1} := (b_{w,\rho})_{w \in T_\rho \setminus \{\rho\}}$ be sub-arrays of B_ρ . Using the induction hypothesis on (2.10), for each $\pi \in S_k$ we have

$$\begin{aligned} P_\rho^\circ(\pi \circ B_{L_1}) &= \left(\frac{\prod_{u \in L_1} (1 - P_u^{(l)}(\pi \circ B_u))}{\sum_{m=1}^k \prod_{u \in L_1} (1 - P_u^{(m)}(\pi \circ B_u))} \right)_{l=1}^k = \left(\frac{\prod_{u \in L_1} (1 - P_u^{(\pi^{-1}(l))}(B_u))}{\sum_{m=1}^k \prod_{u \in L_1} (1 - P_u^{(m)}(B_u))} \right)_{l=1}^k \\ &= \pi^{-1} \circ P_\rho^\circ(B_{L_1}). \end{aligned}$$

Hence set $\{l : (P_\rho^\circ(\tilde{\pi} \circ \pi_\rho^1 \circ B_{L_1}))^{(l)} = \|P_\rho^\circ(B_{L_1})\|_\infty\}$ has the same size for all $\tilde{\pi} \in S_k$ and contains l_ρ if $\tilde{\pi} \in \text{supp } \nu_1(l_\rho)$. Furthermore, by the symmetry of σ_{L_n} , each element of $\{\pi \circ B_\rho\}_{\pi \in S_k}$ is equally likely to happen. Therefore by (2.7), the belief of Bob after the first action of Alice on T_ρ satisfies that

$$\begin{aligned} P_\rho^1 &= P_\rho^1(l_\rho, \pi_\rho^1 \circ B_{L_1}) := \left(P(\sigma_\rho = l \mid l_\rho, \pi_\rho^1 \circ B_{L_1}) \right)_{l=1}^k \\ &= \sum_{\tilde{\pi} \in S_k} \nu_1(l_\rho)(\tilde{\pi}) P_\rho^\circ(\tilde{\pi}^{-1} \circ \pi_\rho^1 \circ B_{L_1}) = \Lambda(P_\rho^\circ), \end{aligned}$$

where the same randomness U_ρ^1 is used in breaking ties of Λ . It follows that $P_\rho^1 \sim (\Lambda \circ \Gamma_s)\mu_k$.

Next we note that for any $\tilde{\pi} \in S_k$, $\|P_\rho^\circ(\tilde{\pi} \circ B_{L_1})\|_\infty = \|P_\rho^\circ(B_{L_1})\|_\infty$. Therefore p_ρ , as a function of $\|P_\rho^\circ(B_{L_1})\|_\infty$ and U_ρ^2 , is invariant under permutations of B_{L_1} . Given $b_{\rho,\rho} = (p_\rho, \eta_{\rho,\rho})$, Bob's posterior estimation of l_ρ and $\pi_\rho^1 \circ B_{L_1}$ satisfies that

$$P(l_\rho = \tilde{\pi}_2^{-1}(\eta_{\rho,\rho}), \pi_\rho^1 \circ B_{L_1} = \tilde{\pi}_2^{-1} \circ B_{\rho,L_1} \mid B_\rho) = \nu_2(p_\rho)(\tilde{\pi}_2).$$

Applying (2.8), we have that

$$P_\rho(B_\rho) = \sum_{\tilde{\pi}_2 \in S_k} \nu_2(p_\rho)(\tilde{\pi}_2) P_\rho^1(\tilde{\pi}_2^{-1}(\eta_{\rho,\rho}), B_{\rho,L_1}) = (1 - p_\rho) P_\rho^1(\eta_{\rho,\rho}, B_{\rho,L_1}) + p_\rho \cdot \left(\frac{1}{k}, \dots, \frac{1}{k} \right).$$

Recall that $p_\rho = \tilde{q}_*(\|P_\rho^\circ\|_\infty, U_\rho^2) = \tilde{q}_*(\|P_\rho^1\|_\infty, U_\rho^2)$ where \tilde{q}_* is the function that reduces $(\Lambda \circ \Gamma_s)\mu_k$ to μ_k and \tilde{q}_* is defined in (2.9). Lemma 2.4 then implies that P_ρ follows the distribution of μ_k .

Finally we finish the induction hypothesis of (2.10). Observe that for $\tilde{\pi} \sim \nu_1(l)$, $\pi \circ \tilde{\pi} \circ \pi^{-1}$ follows the distribution $\nu_1(\pi(l))$. For each $\pi \in S_k$, we have

$$\begin{aligned} P_\rho^1(\pi(l_\rho), \pi(\pi_\rho^1 \circ B_{L_1})) &= \sum_{\tilde{\pi} \in S_k} \nu_1(\pi(l_\rho))(\tilde{\pi}) P_\rho^\circ(\tilde{\pi}^{-1} \circ \pi \circ B_{L_1}) \\ &= \sum_{\tilde{\pi} \in S_k} \nu_1(l_\rho)(\tilde{\pi}) P_\rho^\circ(\pi \circ \tilde{\pi}^{-1} \circ \pi^{-1} \circ \pi \circ B_{L_1}) \\ &= \sum_{\tilde{\pi} \in S_k} \nu_1(l_\rho)(\tilde{\pi}) P_\rho^\circ(\pi \circ \tilde{\pi}^{-1} \circ B_{L_1}) = \pi^{-1} \circ P^1(l_\rho, \pi_\rho^1 \circ B_{L_1}). \end{aligned}$$

It follows that

$$P_\rho(\pi \circ B_\rho) = (1 - p_\rho)P_\rho^1(\pi(\eta_{\rho,\rho}), \pi \circ B_{\rho,L_1}) + p_\rho \cdot \left(\frac{1}{k}, \dots, \frac{1}{k}\right)_\cdot = \pi^{-1} \circ P_\rho(B_\rho).$$

And that finishes the proof the induction hypothesis. \square

Theorem 2.3 and Theorem 2.5 immediately imply the following result.

Corollary 2.6. *For any d, k such that Theorem 2.3 holds, there exist independent random array \mathbf{U} and measurable function $B_\rho(T, \sigma_{L_n}, \mathbf{U})$ such that*

$$\liminf_{n \rightarrow \infty} \mathbb{E} \sup_{l \in [k]} \left| P(\sigma_\rho = l \mid B_\rho(T_n, \sigma_{L_n}, \mathbf{U})) - \frac{1}{k} \right| > 0.$$

2.3 Regular trees

The result of Theorem 2.5 and Corollary 2.6 can be modified to regular trees by, roughly speaking, truncating $T \sim \mathcal{T}_d$ into a smaller tree: Let $\text{tPois}(d', d)$ be the truncated Poisson distribution defined as the distribution of $D' \cdot \mathbf{1}\{D' \leq d\}$ where $D' \sim \text{Pois}(d')$ and let $\mathcal{T}_{\text{tPois}(d', d)}$ be the Galton-Watson tree of offspring distribution $\text{tPois}(d', d)$. There exists a natural coupling between $T_1 \sim \mathcal{T}_{\text{tPois}(d', d)}$, $T_2 \sim \mathcal{T}_{\text{Pois}(d')}$ and $T \sim \mathcal{T}_d$ such that T_1 is a subtree of T_2 and T with probability 1.

Recall that $\mathcal{M}(\Delta^k)$ -operator Γ defined in (2.3) depends implicitly on the offspring distribution ξ . We differentiate the two operators under $\xi = \mathcal{T}_{\text{Pois}(d')}$ and $\xi = \mathcal{T}_{\text{tPois}(d', d)}$ as Γ^p and Γ^t respectively. Fix $\beta^* \in (\beta^0, 1)$. For any d, k satisfying (1.3), let $d' := \lfloor d - (\beta^* - \beta^0)k \rfloor$. For $k \geq k_0(\beta^*, c)$,

$$d_{\text{TV}}(\Lambda \circ \Gamma_s^p \mu_k, \Lambda \circ \Gamma_s^t \mu_k) \leq P(\text{Pois}(d') > d) < c(k \log k)^{-1}. \quad (2.11)$$

Therefore if (d', k) further satisfies Theorem 2.3, then $(\Lambda \circ \Gamma^t)_s \mu_k$ stochastically dominates μ_k . Thus we can find function q_t that reduces $(\Lambda \circ \Gamma^t)_s \mu_k$ to μ_k and define \tilde{q}_t similarly.

Let $T \sim \mathcal{T}_d$ be the n -level d -ary tree and $\mathbf{D} := (D_v)_{v \in T}$ be a T -indexed array of independent $\text{tPois}(d', d)$ random variables. We now describe the necessary modification such that $\tilde{A}_v, \tilde{B}_v, \tilde{P}_v^\circ, \tilde{P}_v$ can be constructed in a similar fashion as A_v, B_v, P_v°, P_v . The construction remains the same for each $v \in L_n$. For each $v \notin L_n$, we proceed with the following changes:

1. In step 2(a), instead of considering all $u \in L_1^v$, Alice now only uses the first D_v vertices and discards the rest. Namely, letting u_1, \dots, u_d be the d offspring of v , she calculates

$$\tilde{P}_v^\circ := \left(\frac{\prod_{i=1}^{D_v} (1 - \tilde{P}_{u_i}^{(l)})}{\sum_{m=1}^k \prod_{i=1}^{D_v} (1 - \tilde{P}_{u_i}^{(m)})} \right)_{l=1}^k,$$

and sets $\tilde{\mathbf{b}}_{w,v} = (\star, \star)$ for each $w \in T_{u_i}, i > D_v$. She then continues to set $\tilde{\mathbf{a}}_v$ and the rest of $\tilde{\mathbf{B}}_v$ using $\tilde{\mathbf{P}}_v^\circ$ and U_v .

2. In step 2(b), instead of setting $p_v = \tilde{q}_\star(\|\mathbf{P}_v^\circ\|_\infty, U_v^2)$, Alice sets $p_v = \tilde{q}_t(\|\tilde{\mathbf{P}}_v^\circ\|_\infty, U_v^2)$.

In short, Bob now has to reconstruct σ_ρ based only on the information $\tilde{\mathbf{B}}_\rho$ of a truncated tree of T sampled from $\mathcal{T}_{\text{Pois}(d', d)}$, as the information on the rest of the vertices are erased and set to (\star, \star) .

Corollary 2.7. *Fix $\beta^\star \in (\beta^0, 1)$. For any d, k such that $(d' := \lfloor d - (\beta^\star - \beta^0)k \rfloor, k)$ satisfies Theorem 2.3 and (2.11), there exist independent random arrays \mathbf{U}, \mathbf{D} and measurable function $\tilde{\mathbf{B}}_\rho(\sigma_{L_n}, \mathbf{U}, \mathbf{D})$ such that*

$$\liminf_{n \rightarrow \infty} \mathbb{E} \sup_{l \in [k]} \left| P\left(\sigma_\rho = l \mid \tilde{\mathbf{B}}_\rho(\sigma_{L_n}, \mathbf{U}, \mathbf{D})\right) - \frac{1}{k} \right| > 0.$$

Proof. By an essentially parallel argument of Theorem 2.5, we can inductively show that $\tilde{\mathbf{P}}_v^\circ$, as a function of $(T, \sigma_{T_v \cap L_n}, \mathbf{U}_v, \mathbf{D}_v)$, follows the distribution of $\Gamma_s^t \mu_k$ and hence $\tilde{\mathbf{P}}_v \sim \mu_k$ for each $v \in T$. Corollary 2.7 then follows immediately. \square

Proof of Theorem 1.1. Let β^0, c be the constant in Theorem 2.3 and β^\star be selected in Corollary 2.7. For any $k \geq k_0$ and d, k satisfying (1.3), they also satisfy the conditions of Theorem 2.3 and Corollary 2.6. Therefore if the k -colouring model on $T \sim \mathcal{T}_{\text{Pois}(d)}$ is not reconstructible for some d, k in the same region, then we must have

$$\limsup_{n \rightarrow \infty} \mathbb{E}_{T \sim \mathcal{T}_{\text{Pois}(d)}} [\text{Var}(\sigma_\rho \mid \mathbf{B}_\rho(T_n, \sigma_{L_n}, \mathbf{D}))] \leq \limsup_{n \rightarrow \infty} \mathbb{E}_{T \sim \mathcal{T}_{\text{Pois}(d)}} [\text{Var}(\sigma_\rho \mid T_n, \sigma_{L_n})] = 0,$$

where the first step follows from the fact that $\mathbf{B}_\rho = \mathbf{B}_\rho(T_n, \sigma_{L_n}, \mathbf{U})$ is independent of σ_ρ given σ_{L_n} . But that conflicts with the result of Corollary 2.6. The same confliction exists with $T \sim \mathcal{T}_d$, $\tilde{\mathbf{B}}_\rho = \tilde{\mathbf{B}}_\rho(\sigma_{L_n}, \mathbf{U}, \mathbf{D})$ and Corollary 2.7. Therefore both models are reconstructible. \square

3 Proof of Theorem 2.3

In this section we prove the stochastic dominance result of Theorem 2.3. In Section 3.1, we first analyse the transformation Γ induced on $\mathcal{M}(\Lambda^k)$ by (2.3) and give a parameterized candidate of μ_k . In the remaining sections, we verify that the candidate does indeed satisfy Theorem 2.3.

3.1 Reformulating the recursion

Recall the notations in the definition of $\Gamma\mu$ in (2.3), where $\mu = \Pi_1 \mu_s$ for some $\mu_s \in \mathcal{M}_s(\Lambda^k)$. For each $l \in [k], 1 \leq i \leq B_l$, let $m_{i,l} := m(\vec{X}_{i,l}, l) := \arg \max_{m \in [k]} \vec{X}_{i,l}^{(m-l+1)}$ be the coordinate

of \vec{X}_{n+1} that contains the largest entry of $\vec{X}_{i,l}$ and draw $m_{i,l}$ from $[k]$ uniformly at random if $\vec{X}_{i,l} = (\frac{1}{k}, \dots, \frac{1}{k})$. Since μ is tilted from some symmetric measure μ_s , similar to (2.4),

$$P(m_{i,l} = m \mid \|\vec{X}_{i,l}\|_\infty = x) = \begin{cases} x & m = l \\ \frac{1-x}{k-1} & m \neq l \end{cases}.$$

Let $\mu^=(dx) := x\mu(dx)$ and $\mu^\neq(dx) := (1-x)\mu(dx)$. The joint distribution of $(\|\vec{X}_{i,l}\|_\infty, m_{i,l})$ satisfies

$$P(\|\vec{X}_{i,l}\|_\infty \in dx, m_{i,l} = m) = \begin{cases} \mu^=(dx) & l = m \\ \frac{1}{k-1}\mu^\neq(dx) & l \neq m \end{cases}, \quad \forall x \in [0, 1], m \in [k].$$

For each $m \in [k]$, define

$$C_m^- := \{(i, m) : m_{i,m} = m\}, \quad C_m^\neq := \{(i, l) : l \neq m, m_{i,l} = m\} \quad \text{and} \quad C_m := C_m^- \cup C_m^\neq.$$

Let c_m^-, c_m^\neq be the cardinality of C_m^- and C_m^\neq respectively and set $p_\neq := \mu^\neq([\frac{1}{k}, 1]) = 1 - \mu^-(\frac{1}{k}, 1]$ to be the probability of $\{(i, l) \notin C_l^-\}$. Note that no offspring of the root has colour 1. Given $d_\rho = \sum_{l=1}^k B_l$, $(c_1^-, c_2^-, \dots, c_k^-, c_1^\neq, c_2^\neq, \dots, c_k^\neq)$ follows multinomial distribution of sum d_ρ and probability

$$\frac{1}{k-1} \left(0, 1 - p_\neq, \dots, 1 - p_\neq, p_\neq, \frac{k-2}{k-1}p_\neq, \dots, \frac{k-2}{k-1}p_\neq \right). \quad (3.1)$$

We now use the new notations to rewrite (2.3). For each $\vec{X}_{i,l} \neq (\frac{1}{k}, \dots, \frac{1}{k})$, the entries of $\vec{X}_{i,l}$ take only two values: $\|\vec{X}_{i,l}\|_\infty$ and $(1 - \|\vec{X}_{i,l}\|_\infty)/(k-1)$. And $\vec{X}_{i,l}^{(m-l+1)} = \|\vec{X}_{i,l}\|_\infty$ if and only if $m = m_{i,l}$. Let $\varphi(x) := \log[(1 - \frac{1-x}{k-1})/(1-x)]$, which is an increasing function mapping $[0, 1]$ to $[-\infty, \infty]$. By taking out the common factor of $\prod_{l,i}(1 - \frac{1-\|\vec{X}_{i,l}\|_\infty}{k-1})$, we rewrite (2.3) as

$$\vec{X}_{n+1}^{(m)} \stackrel{d.}{=} \frac{\prod_{(i,l) \in C_m} (1 - \|\vec{X}_{i,l}\|_\infty)/(1 - \frac{1-\|\vec{X}_{i,l}\|_\infty}{k-1})}{\sum_{m'=1}^k \prod_{(i,l) \in C_{m'}} (1 - \|\vec{X}_{i,l}\|_\infty)/(1 - \frac{1-\|\vec{X}_{i,l}\|_\infty}{k-1})} = \frac{\prod_{(i,l) \in C_m} e^{-\varphi(\|\vec{X}_{i,l}\|_\infty)}}{\sum_{m'=1}^k \prod_{(i,l) \in C_{m'}} e^{-\varphi(\|\vec{X}_{i,l}\|_\infty)}}. \quad (3.2)$$

Note that the exact value of $m_{i,l}$ when $\vec{X}_{i,l} = (\frac{1}{k}, \dots, \frac{1}{k})$ does not matter since $\varphi(\frac{1}{k}) = 0$. We further rewrite (3.2) as

$$\vec{X}_{n+1}^{(m)} \stackrel{d.}{=} \frac{\left(\prod_{i=1}^{c_m^-} \exp(-\varphi(Y_{i,m}^-)) \prod_{i=1}^{c_m^\neq} \exp(-\varphi(Y_{i,m}^\neq)) \right)}{\sum_{l=1}^k \left(\prod_{i=1}^{c_l^-} \exp(-\varphi(Y_{i,l}^-)) \prod_{i=1}^{c_l^\neq} \exp(-\varphi(Y_{i,l}^\neq)) \right)} =: \frac{\exp(-Z_m)}{\sum_{m=1}^k \exp(-Z_m)}. \quad (3.3)$$

where $Y_{i,l}^=$ and $Y_{i,l}^\neq$ are i.i.d. samples of $\frac{1}{1-p^\neq}\mu^=$ and $\frac{1}{p^\neq}\mu^\neq$ respectively and

$$Z_m := \sum_{i=1}^{c_m^=} \varphi(Y_{i,m}^=) + \sum_{i=1}^{c_m^\neq} \varphi(Y_{i,m}^\neq).$$

We conclude our calculation so far in the following claim.

Proposition 3.1. *For any d, k , if there exists $\nu_k \in \mathcal{M}([\frac{1}{k}, 1])$ (with its unique correspondence in $\mathcal{M}(\Lambda^k)$) and $c > 0$, such that $\mu_s = \Pi_1^{-1}(\varphi^{-1} \circ \nu_k) \in \mathcal{M}_s(\Lambda^k)$ and for the $(Z_m)_{m=1}^k$ defined as above using μ_s ,*

$$W := \log \left[\frac{k-2}{k-1} + \frac{1}{\sum_{m=2}^k \exp(Z_1 - Z_m)} \right] \vee 0 \succ_{c/\log k} \nu_k, \quad (3.4)$$

then μ_s satisfies the requirement of Theorem 2.3.

Proof. Maximizing (3.3) over $m \in [k]$, we have that

$$\|\vec{X}_{n+1}\|_\infty = \frac{\max\{1, \exp(Z_1 - Z_m), m=2, \dots, k\}}{1 + \sum_{m=2}^k \exp(Z_1 - Z_m)} \geq \frac{1}{1 + \sum_{m=2}^k \exp(Z_1 - Z_m)} \vee \frac{1}{k}.$$

Composing φ to both side yields that $\varphi(\|\vec{X}_{n+1}\|_\infty) \succ W$. Theorem 2.3 then follows from the fact that $\|\Lambda(\vec{X}_{n+1})\|_\infty = \|\vec{X}_{n+1}\|_\infty$. \square

We now propose a parameterized candidate of ν_k : Let $\delta, \kappa \in (0, 1)$, $M \gg 0$, $0 < \gamma, \alpha_0, \sigma, \epsilon \ll 1$ be parameters to be determined in the order of $(\delta, \kappa, \alpha_0, M, \sigma, \gamma, \epsilon)$ and write $\alpha = \varphi(\frac{1}{2} - \alpha_0) = \log 2 - O(\alpha_0) + o_k(1)$. Let ν_\star be an infinite-volume measure defined as (recalling that $\varphi(\frac{1}{k}) = 0$)

$$\nu_\star(dy) := \kappa \delta_0(dy) + (1 - \kappa) \delta_\alpha(dy) + \frac{\gamma}{y^2} e^{\delta y} \mathbf{1}\{y > M\} dy, \quad (3.5)$$

where δ_x is the Dirac measure at x , and write $\nu_r(dy) := \frac{\gamma}{y^2} e^{\delta y} \mathbf{1}\{y > M\} dy$ for the right tail of ν_\star . We will use ν_\star as a “scaling limit” of ν_k and show that the assumption of Prop. 3.1 is satisfied with

$$\nu_k(dy) := \frac{1}{\log k} \nu_\star(dy) \mathbf{1}\{0 \leq y \leq a_k\},$$

for some choice of $(\delta, \kappa, \alpha_0, M, \sigma, \gamma, \epsilon)$ and $k \geq k_0 = k_0(\delta, \kappa, \alpha_0, M, \sigma, \gamma, \epsilon)$, where a_k is the constant such that ν_k is a probability measure.

For convenience of notation, we will write $k \geq k_0$ where k_0 depends on all six parameters. We will use $1_{\leq a_k}$ or $1_{\geq c_k}$ to cut (part of) a measure above or below such that the total mass is 1. The exact value of a_k and c_k can be derived implicitly and may vary from line to line. Let $\nu_\star^=(dy) := \varphi \circ \mu^=(dx) = \varphi \circ x\mu(dx) = \varphi^{-1}(y)\nu_\star(dy)$, where $\varphi^{-1}(y) = 1 - (e^y + (k-1)^{-1})^{-1}$ and

define $\nu_\star^\neq, \nu_r^=, \nu_r^\neq, \nu_k^=, \nu_k^\neq$ similarly. We define the tail weights

$$\begin{aligned} p_r^\neq &:= \frac{1}{\gamma} \nu_r^\neq([M, \infty)) = \int_M^\infty \frac{e^{\delta y}}{y^2(e^y + \frac{1}{k-1})} dy < \infty \\ p_k^\neq &:= \mu_k^\neq([1/k, 1)) = \nu_k^\neq([0, \infty)) \\ &\leq \frac{1}{\log k} \left[\frac{1}{k}(1 - \kappa) + \left(\frac{1}{2} - \alpha_0\right)\kappa + \gamma p_r^\neq \right] = (1 - o(1)) \frac{\gamma p_r^\neq}{\log k}. \end{aligned}$$

3.2 Distribution of Z_m

In this section we bound the distribution of Z_m in terms of ν_\star . Let $D := d/(k-1) = \log k + \log \log k + \beta$. For $T \sim \mathcal{T}_{\text{Pois}(d)}$, (3.1) implies that $(c_m^=, c_m^\neq)$'s are independent Poisson random variables with rate $(0, p_k^\neq D)$ for $m = 1$ and $((1 - p_k^\neq)D, \frac{k-2}{k-1}p_k^\neq D)$ for $m \geq 2$. Hence, for $m \geq 2$,

$$\begin{aligned} Z_m &\stackrel{d.}{=} \left(\text{Pois}((1 - p_k^\neq)D) \otimes \frac{1}{1 - p_k^\neq} \nu_k^= \right) \oplus \left(\text{Pois}\left(\frac{k-2}{k-1}p_k^\neq D\right) \otimes \frac{1}{p_k^\neq} \nu_k^\neq \right) \\ &= \text{Pois}\left(\left(1 - \frac{p_k^\neq}{k-1}\right)D\right) \otimes \frac{\nu_k^= + \frac{k-2}{k-1}\nu_k^\neq}{(1 - \frac{1}{k-1}p_k^\neq)} \succ \text{Pois}\left(\left(1 - \frac{p_k^\neq}{k-1}\right)D\right) \otimes \frac{\nu_k}{(1 - \frac{1}{k-1}p_k^\neq)} 1_{\leq a_k}, \end{aligned}$$

where the last line follows from that $(\nu_k^= + \frac{k-2}{k-1}\nu_k^\neq)(dy) \leq \nu_k(dy)$. Namely, Z_m stochastically dominates the sum of points in a Poisson point process with intensity $D\nu_k 1_{\leq a_k^0}$, where a_k^0 satisfies $\nu_k([0, a_k^0]) = 1 - \frac{1}{k-1}p_k^\neq$. We expand the summation according to the three parts of ν_k as in (3.5). Firstly, δ_0 does not contribute to the summation. For the second term, we define $S_1 := \text{Pois}(\kappa) \otimes \delta_\alpha$ and note that $\kappa \leq \frac{\kappa D}{\log k}$. Finally for $k \geq k_0$, the total intensity coming from the right tail of ν_k satisfies

$$D\nu_k([M, a_k^0]) = D(\nu_k([0, a_k^0]) - \log^{-1} k) = D - 1 - O(k^{-1} \log k) \geq D - 1 - \gamma.$$

and $(1 - \frac{1}{k-1}p_k^\neq)^{-1} \nu_k(dy) \leq \frac{1+\gamma}{D-1-\gamma} \nu_r(dy)$. Therefore defining

$$S_0 := \text{Pois}(D - 1 - \gamma) \otimes \frac{1 + \gamma}{D - 1 - \gamma} \nu_r 1_{\leq a_k},$$

it follows that $Z_m \succ S_0 + S_1$. We first show the following bound for S_0 .

Lemma 3.2. *For any $M > M(\alpha_0) \vee \frac{2}{\delta}$, there exists constant $C_M > 0$ such that*

$$S_0 \succ \frac{e^{\gamma+1-\beta}}{k \log k} (\delta_0 + (1 + C_M \gamma) \nu_r 1_{\leq a_k^1}), \quad (3.6)$$

where a_k^1 satisfies $1 + (1 + C_M \gamma) \nu_r([M, a_k^1]) = k \log k e^{-(\gamma+1-\beta)}$.

Proof. Let $B_0 \sim \text{Pois}(D-1-\gamma)$ and Y_i be i.i.d. samples of distribution $\frac{1+\gamma}{D-1-\gamma} \nu_r 1_{\leq a_k}$. We have

$$P(S_0 = 0) = P(B_0 = 0) = e^{-(D-1-\gamma)} \leq \frac{1}{k \log k} e^{1+\gamma-\beta}.$$

Since ν_r is supported on $[M, \infty)$ and is absolutely continuous, for $z \geq M$,

$$\begin{aligned} f_{S_0}(z) &= \frac{d}{dz} P\left(\sum_{i=1}^{B_0} Y_i \leq z\right) \leq \sum_{n=1}^{\lfloor z/M \rfloor} P(B_0 = n) \frac{d}{dz} \left[\int_{\sum y_i \leq z} \left(\frac{1+\gamma}{D-1-\gamma}\right)^n \nu_r(dy_1) \cdots \nu_r(dy_n) \right] \\ &\leq \frac{e^{1+\gamma-\beta}}{k \log k} \sum_{n=1}^{\lfloor z/M \rfloor} \frac{1}{n!} \frac{d}{dz} \left[\int_{y_i \geq M, \sum_{i=1}^n y_i \leq z} \frac{(1+\gamma)^n \gamma^n}{y_1^2 y_2^2 \cdots y_n^2} e^{\delta(y_1 + \cdots + y_n)} dy_1 \cdots dy_n \right] \\ &= \frac{e^{1+\gamma-\beta}}{k \log k} \sum_{n=1}^{\lfloor z/M \rfloor} \frac{(1+\gamma)^n \gamma^n}{n!} e^{\delta z} \int_{y_i \geq M, \sum_{i=1}^{n-1} y_i \leq z-M} \frac{1}{y_1^2 \cdots y_{n-1}^2 (z - \sum_{i=1}^{n-1} y_i)^2} dy_1 \cdots dy_{n-1}. \end{aligned}$$

Applying Fact 3.3 below for $n \geq 2$, we have that for $z \geq M$,

$$\begin{aligned} f_{S_0}(z) dz &\leq \frac{e^{1+\gamma-\beta}}{k \log k} \left((1+\gamma)\gamma + \sum_{n=2}^{\infty} \frac{((1+\gamma)\gamma C_M)^n}{n!} \right) \frac{1}{z^2} e^{\delta z} dz \\ &\leq \frac{e^{1+\gamma-\beta}}{k \log k} (1 + C'_M \gamma) \frac{\gamma}{z^2} e^{\delta z} dz = \frac{e^{1+\gamma-\beta}}{k \log k} (1 + C'_M \gamma) \nu_r(dz). \end{aligned}$$

The desired result follows from the last equation and the fact that $P(S_0 \in (0, M)) = 0$. \square

Fact 3.3. *There exist constant C_M such that for $n \geq 2$ and $z \geq nM$,*

$$\int_{y_i \geq M, \sum_{i=1}^{n-1} y_i \leq z-M} \frac{1}{y_1^2 \cdots y_{n-1}^2 (z - \sum_{i=1}^{n-1} y_i)^2} dy_1 \cdots dy_{n-1} \leq \frac{C_M^n}{z^2}.$$

The proof of Fact 3.3 is postponed to Section 4. Next consider the independent sum of $S_0 + S_1$.

Lemma 3.4. *For any $M > M(\alpha_0) \vee \frac{2}{\delta}$ and constant C_M specified in Lemma 3.2,*

$$Z_m \succ S_0 + S_1 \succ \frac{e^{\gamma+1-\beta}}{k \log k} [\nu_{S_1} + (1+\alpha_0)(1+C_M\gamma) \exp(\kappa(e^{-\alpha\delta}-1)) \nu_r 1_{\leq a_k}]. \quad (3.7)$$

Proof. Letting $\nu_{S_0}^+ := \nu_r 1_{\leq a_k^1}$ where a_k^1 is defined in (3.6), we have

$$\nu_{S_0+S_1} = \frac{e^{1+\gamma-\beta}}{k \log k} (\delta_0 * \nu_{S_1} + (1+C_M\gamma) \nu_{S_0}^+ * \nu_{S_1}) = \frac{e^{1+\gamma-\beta}}{k \log k} (\nu_{S_1} + (1+C_M\gamma) \nu_{S_0}^+ * \nu_{S_1}). \quad (3.8)$$

It is left to verify that $\nu_{S_0+S_1}^+ := \nu_{S_0}^+ * \nu_{S_1} \succ (1+\alpha_0) \exp(\kappa(e^{-\alpha\delta}-1)) \nu_r 1_{\leq a_k}$ where a_k is chosen such that RHS of (3.8) has total mass 1. Recall that $S_1 \stackrel{d}{=} \alpha \cdot \text{Pois}(\kappa)$. $\nu_{S_0+S_1}^+$ is absolutely continuous

and supported on $[M, \infty)$. For $z \geq M$ we have

$$f_{S_0+S_1}^+(z) = \sum_{n=0}^{\infty} \frac{\kappa^n e^{-\kappa}}{n!} f_{S_0}^+(z - n\alpha) \leq \sum_{n=0}^{\infty} \frac{\kappa^n e^{-\kappa}}{n!} \frac{\gamma e^{\delta(z-n\alpha)}}{(z-n\alpha)^2} \mathbf{1}\{z - n\alpha \geq M\}$$

To control the $(z - n\alpha)^{-2}$ term, we first choose for any $\alpha > 0$ a $N = N(\alpha_0)$ such that $\sum_{n=N+1}^{\infty} \frac{1}{n!} \leq \frac{1}{2e} \alpha_0$ and then choose $M(\alpha_0)$ such that for $M > M(\alpha_0)$, $n \leq N$ and $z \geq M$,

$$(1 - n\alpha/z)^{-2} \leq (1 - n\alpha/z)^{-2} \leq 1 + \alpha_0/2. \quad (3.9)$$

Observe that $\frac{\gamma}{z^2} e^{\delta z}$ is monotone increasing for $z \in (\frac{2}{\delta}, \infty)$. For all $M > M(\alpha_0) \vee \frac{2}{\delta}$ and $z \geq M$,

$$\begin{aligned} f_{S_0+S_1}^+(z) &\leq \frac{\gamma e^{\delta z}}{z^2} \sum_{n=0}^N \frac{\kappa^n e^{-\kappa}}{n!} \frac{e^{-n(\alpha\delta)}}{(1 - n\alpha/z)^2} + \frac{\gamma e^{\delta z}}{z^2} \sum_{n=N+1}^{\infty} \frac{\kappa^n e^{-\kappa}}{n!} \\ &\leq (1 + \alpha_0) \exp[\kappa(e^{-\alpha\delta} - 1)] \nu_r(dz). \end{aligned}$$

The proof finishes by cutting ν_r at the place such that (3.8) has the total mass 1. \square

Finally, for $m = 1$ and $k \geq k_0$ such that $\frac{D}{\log k} \leq (1 + \gamma) \vee (1 + \alpha_0)$, we have

$$Z_1 \stackrel{d}{=} \text{Pois}(p_k^\# D) \otimes \frac{1}{p_k^\#} \nu_k^\# \prec \left(\text{Pois}\left(\frac{1}{2}\kappa\right) \otimes \delta_\alpha \right) \oplus \left(\text{Pois}(\gamma p_r^\#) \otimes \frac{1}{\gamma p_r^\#} \nu_r^\# \right), \quad (3.10)$$

where the second term is 0 with probability $\exp(-\gamma p_r^\#)$.

3.3 Distribution of $\sum_{m=2}^k \exp(-Z_m)$

In this section we analysis the distribution of $\sum_{m=2}^k \exp(-Z_m) = (k-1) \otimes \exp(-Z_m)$. Let $\psi(x) := e^{-x}$. An easy calculation gives that

$$\psi \circ \nu_\star(dx) = (1 - \kappa) \delta_1(dx) + \kappa \delta_{\psi(\alpha)}(dx) + \frac{\gamma}{(\log x)^2} \frac{1}{x^{1+\delta}} \mathbf{1}\{0 < x < \psi(M)\} dx.$$

Define

$$C_Z := C_Z(\delta, \kappa, \alpha_0, M, \gamma) = (1 + \alpha_0)(1 + C_M \gamma) \exp(\kappa(e^{-\alpha\delta} - 1)). \quad (3.11)$$

Now (3.7) can be rewritten as

$$\psi(Z_m) \prec \frac{1}{k \log k} e^{\gamma+1-\beta} \left[\psi \circ \nu_{S_1} + C_Z \frac{\gamma}{(\log x)^2} \frac{1}{x^{1+\delta}} \mathbf{1}\{c_k < x < \psi(M)\} \right]. \quad (3.12)$$

As k grows, the density of $\psi(Z_m)$ diverges quickly around 0 and the probability of seeing $Z_m \geq x$ for more than one $m \in [k]$ is $o(\frac{1}{k})$ for any fixed $x > 0$. Hence intuitively,

$$\nu_{k \otimes \psi(Z_m)} \approx \nu_{\max_{m \in [k]} \psi(Z_m)} \approx k \cdot \nu_{\psi(Z_m)}.$$

Lemma 3.5. Fix $\delta = \frac{1}{2}$. For any $M > M(\alpha_0) \vee \frac{2}{\delta}$ such that (3.8) holds and $\sigma, \epsilon > 0$, $k \geq k_0$,

$$(k-1) \otimes \psi(Z_m) \prec \frac{e^{\gamma+1-\beta}}{\log k} \left[(\psi + \sigma) \circ \nu_{S_1} + (1+\epsilon) C_Z \frac{\gamma}{(\log x)^2} \frac{1}{x^{1+\delta}} \mathbf{1}_{x \leq \psi(M)} \right] \mathbf{1}_{\geq c_k} + \frac{\epsilon}{\log k} \delta_\infty,$$

where $(\psi + \sigma)(x) := \psi(x) + \sigma$ and C_Z is defined in (3.11).

Proof. We recall the RHS of (3.12) and treat its discrete part and continuous part separately. Let $p_1 := \frac{e^{\gamma+1-\beta}}{k \log k}$, $\mu_Z^1 := \psi \circ \nu_{S_1}$ and $\mu_Z^2(dx) := \frac{p_1}{1-p_1} \frac{\gamma}{(\log x)^2} x^{-(1+\delta)} \mathbf{1}_{c_k < x \leq \psi(M)} dx$. Among the $(k-1)$ i.i.d. samples from the RHS of (3.12), $b \sim \text{Binom}(k-1, p_1)$ of them comes from μ_Z^1 and the rest comes from μ_Z^2 . Choose $C_b > 0$ such that for any $k \geq k_0$, $P(b \geq 2) \leq C_b \log^{-2} k$. It follows that

$$\begin{aligned} (k-1) \otimes \psi(Z_m) &\prec (\text{Binom}(k, p_1) \otimes \mu_Z^1) \oplus (k \otimes \mu_Z^2) \\ &\prec \left[(1 - kp_1) \cdot k \otimes \mu_Z^2 + kp_1 \cdot (\mu_Z^1 \oplus (k \otimes \mu_Z^2)) \right] \mathbf{1}_{\geq c_k} + \frac{C_b}{\log^2 k} \delta_\infty. \end{aligned} \quad (3.13)$$

We will show in Lemma 3.8 that for any $\epsilon > 0$ and $k \geq k_0$,

$$k \otimes \mu_Z^2 \prec (1+\epsilon)k \cdot \mu_Z^2 \mathbf{1}_{\geq c_k^0} + \frac{\epsilon}{2 \log k} \delta_\infty. \quad (3.14)$$

Therefore for any $\sigma > 0$, there exists $C_\sigma > 0$ such that for $k \geq k_0$, $P(k \otimes \mu_Z^2 \geq \sigma) \leq C_\sigma \log^{-1} k$ and

$$\begin{aligned} \text{RHS of (3.13)} &\prec \left[(1 - kp_1) \cdot k \otimes \mu_Z^2 + kp_1 \cdot (\mu_Z^1 * \delta_\sigma) + \frac{kp_1 C_\sigma}{\log k} \cdot \delta_\infty \right] \mathbf{1}_{\geq c_k} + \frac{C_b}{\log^2 k} \delta_\infty \\ &\prec \left[(1+\epsilon)k(1 - kp_1) \cdot \mu_Z^2 \mathbf{1}_{\geq c_k^0} + kp_1 \cdot (\mu_Z^1 * \delta_\sigma) \right] \mathbf{1}_{\geq c_k} + \frac{\epsilon}{\log k} \delta_\infty \\ &\prec \frac{e^{\gamma+1-\beta}}{\log k} \left[(\psi + \sigma) \circ \nu_{S_1} + (1+\epsilon) C_Z \frac{\gamma}{(\log x)^2} \frac{1}{x^{1+\delta}} \right] \mathbf{1}_{\geq c_k} + \frac{\epsilon}{\log k} \delta_\infty. \end{aligned}$$

where in the last step, we observe that removing the $\mathbf{1}_{\geq c_k^0}$ after μ_Z^2 will only make the measure inside the square bracket stochastically larger after cutting from below. \square

In the remaining of the section, we check that (3.14) is true. We will henceforth omit the $O(1)$ factor $(k \log k) \cdot \frac{p_1}{1-p_1}$ by absorbing it into γ and let

$$U \sim \mu_U := \mu_Z^2 = \frac{1}{k \log k} \frac{\gamma}{(\log x)^2} x^{-(1+\delta)} \mathbf{1}_{\{c_k < x \leq \psi(M)\}} dx. \quad (3.15)$$

Measure μ_U resembles distributions that converge to stable law. However, we can not directly apply the usual proof of convergence for stable laws (cf. Section 3.7 of [11], or the reference there) to $k \otimes U$, since the expression of μ_U also depends on k . With some modification, we show the following result.

Lemma 3.6. *For any $\delta, \gamma \in (0, 1)$, $M > \frac{2}{\delta}$, let $t_k := \inf\{t : \mu_U([t, \infty)) < 1/k\}$, then $k \otimes (t_k^{-1}U)$ converges weakly to the stable law with index δ and characteristic function*

$$\exp\{-b_\star |t|^\delta (1 + i \operatorname{sgn}(t) \tan(\pi\delta/2))\},$$

where sgn is the sign function and $b_\star = \delta \int_0^\infty (\cos x - 1)x^{-(1+\delta)} dx = -\cos(\frac{\pi}{2}\delta)\Gamma(1-\delta)$.

In the proof we use the following calculus result, the proof of which is deferred to Section 4.

Fact 3.7. *Let t_k be defined as in Lemma 3.6, we have*

1. $t_k = (1 + o_k(1))(\frac{\gamma^\delta}{\log k (\log \log k)^2})^{1/\delta}$ and therefore

$$\frac{\gamma}{\delta} t_k^{-\delta} \log^{-2} t_k = (1 + o_k(1)) \log k.$$

2. For any constant $c > 0$,

$$\begin{aligned} \lim_{k \rightarrow \infty} k P(U \geq ct_k) &= \lim_{k \rightarrow \infty} t_k^{-1} \int_{ct_k}^\infty \frac{1}{k \log k} \frac{\gamma}{\log^2 x} \frac{1}{x^{1+\delta}} dx = c^{-\delta}, \\ \lim_{k \rightarrow \infty} k \mathbb{E}(t_k^{-1} U 1_{U \leq ct_k}) &= \lim_{k \rightarrow \infty} t_k^{-1} k \int_0^{ct_k} \frac{1}{k \log k} \frac{\gamma}{\log^2 x} \frac{x}{x^{1+\delta}} dx = c^{1-\delta} \frac{\delta}{1-\delta}, \\ \lim_{k \rightarrow \infty} k \mathbb{E}(t_k^{-2} U^2 1_{U \leq ct_k}) &= \lim_{k \rightarrow \infty} t_k^{-2} k \int_0^{ct_k} \frac{1}{k \log k} \frac{\gamma}{\log^2 x} \frac{x^2}{x^{1+\delta}} dx = c^{2-\delta} \frac{\delta}{2-\delta}. \end{aligned}$$

Proof of Lemma 3.6. Let $U_i, i = 1, 2, \dots, k$ be i.i.d. copies of U and let $S_k := \sum_{i=1}^k U_i$. Given $\omega \in (0, 1)$, let $m_{\leq \omega} := \mathbb{E}(U 1_{\{U \leq \omega t_k\}})$, $S_k^\omega := \sum_{i=1}^k U_i 1_{\{U_i \geq \omega t_k\}}$ and $T_k^\omega := \sum_{i=1}^k U_i 1_{\{U_i < \omega t_k\}} - k m_{\leq \omega}$. We have

$$S_k = S_k^\omega + T_k^\omega + k \cdot m_{\leq \omega}.$$

For the first term S_k^ω , let F_k^ω and ψ_k^ω be the c.d.f. and characteristic function of $t_k^{-1}U_i$ conditioned on $\{t_k^{-1}U_i \geq \omega\}$. By Fact 3.7(2), for any $\omega > 0$ and any $x > \omega$,

$$1 - F_k^\omega(x) = (1 + o_k(1))(x/\omega)^{-\delta} \rightarrow (\omega/x)^\delta, \quad \text{as } k \rightarrow \infty.$$

Hence for any $t \in \mathbb{R}$, $\psi_k^\omega(t) \rightarrow \psi^\omega(t) := \int_\omega^\infty e^{itx} \cdot \delta \omega^\delta x^{-(\delta+1)} dx$. Meanwhile by Fact 3.7(2), the

distribution of the number of $i \in [k]$ such that $U_i \geq \omega t_k$ converges weakly to $\text{Pois}(\omega^{-\delta})$, hence

$$\lim_{k \rightarrow \infty} \mathbb{E} \exp(it S_k^\omega / t_k) = \exp[-\omega^{-\delta}(1 - \psi^\omega(t))] = \exp\left(\int_{\omega}^{\infty} (e^{itx} - 1) \delta x^{-(\delta+1)} dx\right).$$

For the second term T_k^ω , observe that $\mathbb{E} T_k^\omega = 0$. By Fact 3.7,

$$t_k^{-2} \mathbb{E} (T_k^\omega)^2 = t_k^{-2} \text{Var}(T_k^\omega) \leq k t_k^{-2} \mathbb{E} U_i^2 1\{U_i < \omega t_k\} \leq (1 + o_k(1)) \frac{\delta}{2 - \delta} \omega^{2-\delta}.$$

For each $t \in \mathbb{R}$, $\exp(itx)$ is a Lipschitz function with Lipschitz constant t . By Jensen's inequality,

$$|\mathbb{E} \exp(it(t_k^{-1} S_k)) - \mathbb{E} \exp(it(t_k^{-1} S_k^\omega))| \leq t (\mathbb{E} |t_k^{-1} T_k^\omega| + t_k^{-1} k m_{\leq \omega}) \leq O(\omega^{1-\delta/2}).$$

Let $\omega \rightarrow 0$. By dominated convergence theorem, we have

$$\lim_{k \rightarrow \infty} \mathbb{E}(\exp(it S_k / t_k)) = \exp\left(\int_0^{\infty} (e^{itx} - 1) \delta x^{-(\delta+1)} dx\right).$$

The rest of the proof follows from complex analysis: Let Γ denote the gamma function (not to be confused with the recursion Γ_s defined before). For $t > 0$, (the case of $t < 0$ is parallel)

$$\begin{aligned} \int_0^{\infty} (e^{itx} - 1) \delta x^{-(\delta+1)} dx &= t^\delta \int_0^{\infty} (e^{ix} - 1) \delta x^{-(1+\delta)} dx \\ &= i t^\delta \int_0^{\infty} x^{-\delta} e^{ix} dx = i^\delta t^\delta \int_0^{\infty} (ix)^{-\delta} e^{ix} d(ix) \\ &= \Gamma(1 - \delta) i^\delta t^\delta = \cos(\pi\delta/2) \Gamma(1 - \delta) t^\delta (1 + i \tan(\pi\delta/2)), \end{aligned}$$

where the second equality follows by integration by part and the last equality follows by doing contour integral on region $\{re^{i\theta} : \omega \leq r \leq R, \theta \in [0, \frac{\pi}{2}]\}$ and letting $\omega \rightarrow 0, R \rightarrow \infty$. \square

Let \tilde{U} denote the limiting stable law specified in Lemma 3.6. When $\delta = \frac{1}{2}$, \tilde{U} follows the Levy distribution with parameter $\frac{\pi}{2}$. Since this is the only value of δ for which we have a closed formula for $f_{\tilde{U}}$, here and henceforth we will take $\delta = 1/2$. The result, however, should hold for all $\delta \leq \frac{1}{2}$ as long as (3.16) holds. Plugging in the formula of Levy distribution and comparing with Fact 3.7, we have

$$\begin{aligned} P(\tilde{U} \leq c) &= \frac{2}{\sqrt{\pi}} \int_{\frac{1}{2}\sqrt{\pi/c}}^{\infty} e^{-t^2} dt \leq \frac{2}{\sqrt{\pi}} \frac{1}{2} \sqrt{\frac{\pi}{c}} e^{-\pi/2c} \leq c^{-1/2} e^{-\pi/2c} \\ &< c^{-1/2} = (1 + o_k(1)) k P(U < ct_k). \end{aligned} \tag{3.16}$$

Thus we can upper-bound $\mu_{k \otimes U}(dx)$ by $(1 + o_k(1))k \cdot \mu_U(dx)$ for small $x \approx O(t_k)$. In the next

lemma, we bound larger values of $k \otimes U$ using the intuition of $k \otimes U \approx \max_{i=1, \dots, k} U_i$.

Lemma 3.8. *Fix $\delta = 1/2$. For any $M \geq \frac{2}{\delta}$, $\gamma, \epsilon \in (0, 1)$, and $k \geq k_0$,*

$$k \otimes \mu_U \prec (1 + \epsilon)k \cdot \mu_U 1_{\geq c_k} + \frac{\epsilon}{\log k} \delta_\infty. \quad (3.17)$$

Proof. Let U_1, \dots, U_k be i.i.d. copies of U and define $U_{(1)} := \max_{i=1, \dots, k} U_i$, $U_R := \sum_{i=1}^k U_i - U_{(1)}$. Let $c = c(\delta, M, \gamma, \epsilon) > 0$ be some small constant to be determined. We write

$$P\left(\sum_{i=1}^k U_i \geq z\right) \leq P(U_{(1)} \geq (1-c)z) + \int_0^{(1-c)z} f_{U_{(1)}}(x) P(U_R \geq z-x \mid U_{(1)} = x) dx, \quad (3.18)$$

where $f_{U_{(1)}}(z) = k f_U(z) (F_U(z))^{k-1} \leq k f_U(z)$. Fix $\sigma = \sigma(\delta, M, \gamma, \epsilon) \in (0, \frac{1}{2})$ such that

$$P(U \geq (1-\sigma)\psi(M)) \leq \frac{1}{\log k} \int_{(1-\sigma)\psi(M)}^{\psi(M)} \frac{\gamma}{\log^2 x} x^{-(1+\delta)} dx \leq \frac{\epsilon}{2 \log k}.$$

We will split the proof into three cases: $x \in [c_k, Nt_k]$, $x \in [Nt_k, (1-\sigma)\psi(M)]$ and $x \geq (1-\sigma)\psi(M)$ where $N = N(\delta, M, \gamma, \epsilon, \sigma, c)$ is a large constant to be determined.

1. $x \in [Nt_k, (1-\sigma)\psi(M)]$: To bound the first term of (3.18), we observe that f_U is a decreasing function and for $z \leq (1-\sigma)\psi(M)$, $(1+\sigma)z \leq \psi(M) \in \text{supp } U$. Therefore

$$\frac{P(U_{(1)} \in [(1-c)z, z])}{P(U_{(1)} \in [z, (1+\sigma)z])} \leq \frac{cz f_U((1-c)z) F^{k-1}(z)}{\sigma z f_U((1+\sigma)z) F^{k-1}(z)} \leq \frac{c}{\sigma} \frac{f_U(z/2)}{f_U((1+\sigma)z)} \leq C_{\sigma, M} \cdot c,$$

for all $c \leq 1/2$ and $z \leq (1-\sigma)\psi(M)$. It follows that

$$P(U_{(1)} \geq (1-c)z) \leq (1 + C_{\sigma, M} \cdot c) P(U_{(1)} \geq z) \leq (1 + C_{\sigma, M} \cdot c) k P(U \geq z). \quad (3.19)$$

For the second term of (3.18), a similar calculation of Fact 3.7 gives that for any $x \leq \psi(M)$,

$$\begin{aligned} k \log k \mathbb{E}(U \mid U \leq x) &= \frac{k \log k}{F_U(x)} \int_0^x z f_U(z) dz \leq \frac{\gamma}{1-\delta} \frac{1}{F_U(x)} \frac{1}{\log^2 x} x^{1-\delta}, \\ k \log k \mathbb{E}(U^2 \mid U \leq x) &= \frac{k \log k}{F_U(x)} \int_0^x z^2 f_U(z) dz \leq \frac{\gamma}{2-\delta} \frac{1}{F_U(x)} \frac{1}{\log^2 x} x^{2-\delta}. \end{aligned}$$

Recall the expression of t_k from Fact 3.7. For any $c > 0$ we choose $N = N(M, \gamma, \epsilon, c)$ such that for $k \geq k_0$ and $x \geq Nt_k$,

$$k \mathbb{E}(U \mid U \leq x) \leq \frac{1 + o_k(1)}{\log k} \frac{\gamma}{1-\delta} \frac{x(Nt_k)^{-\delta}}{\log^2 t_k} = (1 + o_k(1)) N^{-\delta} \frac{\delta}{1-\delta} x \leq \frac{1}{2} cx. \quad (3.20)$$

Given $U_{(1)} = x$, U_R is distributed as the sum of $(k-1)$ i.i.d. copies of U conditioned on $U \leq x$. By Chebyshev inequality, for any $z \in [2Nt_k, \psi(M)]$ and $x \leq (1-c)z$,

$$P(U_R \geq z - x \mid U_{(1)} = x) \leq \frac{k \cdot \mathbb{E}(U^2 \mid U \leq x)}{(z - x - k \mathbb{E}(U \mid U \leq x))^2} \leq \frac{4}{c^2 z^2} \frac{1}{\log k} \frac{\gamma}{2 - \delta} \frac{1}{F_U(x)} \frac{x^{2-\delta}}{\log^2 x},$$

where in the second step, we use the fact that $\mathbb{E}(U \mid U \leq x)$ is monotone decreasing in x . Plugging the estimation into the RHS of (3.18), for $z \leq \psi(M)$, we have that

$$\begin{aligned} & \int_0^{(1-c)z} k f_U(x) F_U(x)^{k-1} P(U_R \geq z - x \mid U_{(1)} = x) dx \\ & \leq \int_{c_k}^{(1-c)z} \frac{1}{\log k} \frac{\gamma}{\log^2 x} x^{-1-\delta} \cdot \frac{4}{c^2 z^2} \frac{1}{\log k} \frac{\gamma}{2 - \delta} \frac{x^{2-\delta}}{\log^2 x} dx \\ & \leq \frac{C_{c,\gamma}}{\log^2 k} \frac{1}{z^2} \int_{c_k}^{(1-c)z} \frac{1}{\log^4 x} x^{1-2\delta} dx \leq \frac{C_{c,\gamma,M}}{\log^2 k \cdot z^{2\delta} \log^4 z}. \end{aligned} \quad (3.21)$$

Meanwhile, for $z \leq (1-\sigma)\psi(M)$,

$$kP(U \geq z) \geq k \cdot \sigma z f_U((1+\sigma)z) = \frac{C_{\gamma,\sigma,M}}{\log k \cdot z^\delta \log^2 z}. \quad (3.22)$$

Comparing (3.21) and (3.22) and using Fact 3.7(1), we have for all $z \geq Nt_k$ that

$$\int_0^{(1-c)z} f_{U_{(1)}}(x) P\left(\sum_{i=1}^n U_i \geq z \mid U_{(1)} = x\right) dx \leq C_{c,\gamma,\sigma,M} N^{-\delta} k P(U \geq z). \quad (3.23)$$

Combine (3.19) and (3.23). For each $\epsilon > 0$, we can first pick $c \leq \epsilon/2C_{\sigma,M}$ and then choose $N = N(M, \gamma, \epsilon, \sigma, c)$ such that (3.20) is true and for all $k \geq k_0$, $z \in [Nt_k, (1-\sigma)\psi(M)]$,

$$P\left(\sum_{i=1}^k U_i \geq z\right) \leq kP(U \geq z) \left(1 + C_{\sigma,M} \cdot c + \frac{C_{c,\gamma,\sigma,M}}{N^\delta}\right) \leq (1+\epsilon)kP(U \geq z). \quad (3.24)$$

2. $z \in [c_k, Nt_k]$: Lemma 3.6 implies that for $z' \in (1, N]$, $P(\sum_{i=0}^k U_i \geq z't_k)$ converges uniformly to $1 \wedge P(\tilde{U} > z')$ as $k \rightarrow \infty$ and \tilde{U} follows the Levy distribution with parameter $\frac{\pi}{2}$. Comparing the c_k in the RHS of (3.17) to the definition of t_k yields that $c_k \geq t_k$ for any $\epsilon > 0$. Therefore for $k > k_0$ and $z \in [c_k, Nt_k]$ with $z' = z/t_k \in (1, N]$,

$$P\left(\sum_{i=1}^k U_i \geq z\right) \leq (1+\epsilon/2)P(\tilde{U} > z') \leq (1+\epsilon)kP(U \geq z't_k), \quad (3.25)$$

where the last step uses (3.16).

3. Finally using (3.24) and recall the definition of σ , we have for all $z \geq (1 - \sigma)\psi(M)$ that

$$P\left(\sum_{i=1}^k U_i \geq z\right) \leq P\left(\sum_{i=1}^k U_i \geq (1 - \sigma)\psi(M)\right) \leq (1 + \epsilon)kP(U \geq (1 - \sigma)\psi(M)) \leq \frac{\epsilon}{\log k}. \quad (3.26)$$

Combining (3.24), (3.25) and (3.26) completes the proof. \square

3.4 Distribution of $\log(\sum_{m=2}^k \exp(Z_1 - Z_m))$

In this section we bound the distribution of $W_0 := -\log(\sum_{m=2}^k e^{Z_1 - Z_m})$. First we rewrite (3.10) as

$$Z_1 \prec \left(\text{Pois}\left(\frac{1}{2}\kappa\right) \otimes \delta_\alpha\right) \oplus \left(\text{Pois}(\gamma p_r^\neq) \otimes \frac{1}{\gamma p_r^\neq} \nu_r^\neq\right) =: R_0 + R_r =: \tilde{Z}_1,$$

and let $\tilde{\nu}_{-Z_1}$ be the distribution of $-\tilde{Z}_1$. Then we define $V := -\log(\sum_{m=2}^k e^{-Z_m})$. The conclusion of Lemma 3.5 can be rewritten as

$$\begin{aligned} \nu_V &\succ \frac{e^{\gamma+1-\beta}}{\log k} [\psi^{-1} \circ (\psi + \sigma) \circ \nu_{S_1} + (1 + \epsilon)C_Z \nu_r] 1_{\leq a_k} + \frac{\epsilon}{\log k} \delta_{-\infty} \\ &=: \tilde{\nu}_V^1 + \tilde{\nu}_V^r + \tilde{\nu}_V^\infty =: \tilde{\nu}_V. \end{aligned} \quad (3.27)$$

Let \tilde{V} be sampled from $\tilde{\nu}_V$. Note that Z_1 is independent of $\sum_{m=2}^k Z_m$. We finally define

$$\tilde{W}_0 := \tilde{V} - \tilde{Z}_1 \prec V - Z_1 = W_0, \quad (3.28)$$

Lemma 3.9. *Assume that $(\delta, \kappa, \alpha_0, M, \sigma, \gamma, \epsilon)$ satisfies the conditions of Lemma 3.4 and 3.5.*

1. *If $\delta \leq \frac{1}{2}$, then there exists constant $C_{\delta, M} > 0$ such that for each $y \geq M$,*

$$(\nu_r * \tilde{\nu}_{-Z_1})(dy) \leq (1 + C_{\delta, M} \gamma) \exp(\kappa(e^{\alpha\delta} - 1)/2) \nu_r(dy).$$

2. *There exists constant $C_{\delta, \alpha, M}^* > 0$ such that $(\nu_r * \tilde{\nu}_{-Z_1})((-\infty, M]) \leq \gamma \cdot C_{\delta, \alpha, M}^*$.*

3. *For any fixed κ, α_0 and $y_1, y_2 \geq M$,*

$$\liminf_{\sigma, \gamma \rightarrow 0} (\tilde{\nu}_V^1 * \tilde{\nu}_{-Z_1})([y_1, y_2]) \geq \frac{e^{-\kappa/2}}{2 \log k} P(\text{Pois}(\kappa) \cdot \alpha \in (y_1, y_2)).$$

Proof. Part 1: By definition, for any $y \geq M$

$$\nu_r * \tilde{\nu}_{-Z_1}(dy) = \int_{-\infty}^0 \frac{\gamma e^{\delta(y-z)}}{(y-z)^2} \tilde{\nu}_{-Z_1}(dz) \leq \frac{\gamma e^{\delta y}}{y^2} dy \cdot \int_{-\infty}^0 e^{-\delta z} \tilde{\nu}_{-Z_1}(dz) = \nu_r(dy) \mathbb{E} e^{\delta \tilde{Z}_1}. \quad (3.29)$$

Hence it is enough to bound $\mathbb{E} \exp(\delta \tilde{Z}_1) = \mathbb{E} \exp(\delta R_0) \mathbb{E} \exp(\delta R_r)$. For the first term,

$$\mathbb{E} \exp(\delta R_0) = \mathbb{E} \exp(\delta \alpha \cdot \text{Pois}(\kappa/2)) = \exp(\kappa(e^{\alpha\delta} - 1)/2). \quad (3.30)$$

For the second term, R_r has the same distribution as the sum of points from the Poisson point process with intensity $\nu_r^\neq(dy)$. Recall that

$$\nu_r^\neq(dy) = (e^y + (k-1)^{-1})^{-1} \nu_r(dy) \leq \frac{\gamma}{y^2} e^{(\delta-1)y} dy$$

and $p_r^\neq = \frac{1}{\gamma} \nu_r^\neq([M, \infty))$ depends only on δ, M . By Campbell's Theorem, for any $\delta \leq \frac{1}{2}$ and $\gamma \leq 1$,

$$\mathbb{E} \exp(\delta R_r) = \exp\left(\int_M^{a_k} (e^{\delta y} - 1) \nu_r^\neq(dz)\right) \leq \exp\left(\gamma \int_M^\infty y^{-2} e^{(2\delta-1)y} dy\right) \leq 1 + \gamma C_{\delta, M}, \quad (3.31)$$

where in the last step we use the inequality $e^x \leq 1 + xe^x, \forall x \geq 0$. Plugging (3.30) and (3.31) back into (3.29) yields the desired result.

Part 2: Expanding the convolution of $\nu_r * \tilde{\nu}_{-Z_1}$ yields that

$$\nu_r * \tilde{\nu}_{-Z_1}((-\infty, M]) \leq \int_0^\infty \int_M^{z+M} \frac{\gamma}{y^2} e^{\delta y} \cdot \tilde{\nu}_{Z_1}(dz) dy \leq \frac{\gamma e^{\delta M}}{\delta M^2} \int_0^\infty e^{\delta z} \tilde{\nu}_{Z_1}(dz) = \frac{\gamma e^{\delta M}}{\delta M^2} \mathbb{E} e^{\delta \tilde{Z}_1}.$$

Applying (3.30) and (3.31) to $\mathbb{E} e^{\delta \tilde{Z}_1}$ gives one possible $C_{\delta, \alpha, M}^* = \exp(e^{\alpha\delta} - 1)(1 + C_{\delta, M})e^{\delta M}/(\delta M^2)$.

Part 3: Noting that $\psi^{-1}(\psi(y) + \sigma) = -\log(e^{-y} - \sigma)$, we have that

$$\begin{aligned} \tilde{\nu}_V^1 * \tilde{\nu}_{-Z_1}([y_1, y_2]) &\geq \frac{e^{\gamma+1-\beta}}{\log k} P(\tilde{Z}_1 = 0) \cdot P(\text{Pois}(\kappa) \cdot \alpha \in [\log(e^{-y_1} - \sigma), \log(e^{-y_2} - \sigma)]) \\ &\geq \frac{1}{\log k} e^{-\frac{1}{2}\kappa - \gamma p_r^\neq} P(\text{Pois}(\kappa) \cdot \alpha \in (\log(-e^{-y_1} - \sigma), -\log(e^{-y_2} - \sigma))). \end{aligned}$$

$\text{Pois}(\kappa) \cdot \alpha$ takes values from the discrete set $\alpha\mathbb{Z}_+$. For any fixed y_1, y_2 , there exists $\sigma = \sigma(\alpha, y_1, y_2)$ such that there is no points of $\alpha\mathbb{Z}_+$ between $-\log(e^{-y_i} - \sigma)$ and y_i , $i = 1, 2$. Hence in the last line we can substitute the probability by $P(\text{Pois}(\kappa) \cdot \alpha \in (y_1, y_2))$. Letting $\gamma \rightarrow 0$ finishes the proof. \square

3.5 Final step

Finally we are ready to prove Theorem 2.3.

Proof of Theorem 2.3. By Proposition 3.1, it suffices to show that under certain choice of parameters $(\delta, \kappa, \alpha_0, M, \sigma, \gamma, \epsilon)$, the random variable W defined in (3.4) stochastically dominates ν_k by $c/\log k$ for some fixed $c > 0$. For any $\alpha_0 > 0$ and $\alpha = \varphi(\frac{1}{2} - \alpha_0)$, we first choose $\sigma < \sigma_1(\alpha_0)$ such

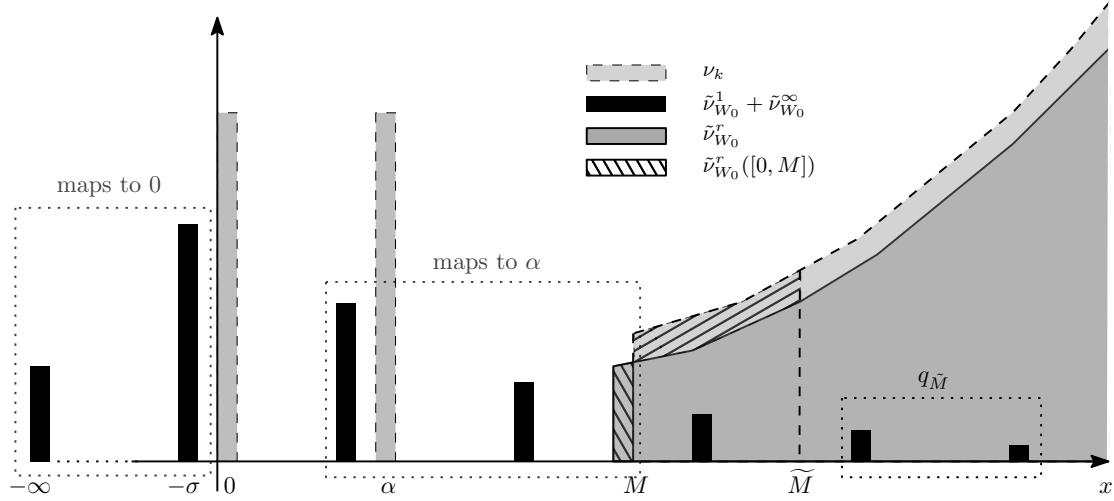


Figure 3.1: $\tilde{\nu}_{W_0}$ and ν_k

that $\log(1 + e^{-\sigma}) > \frac{1}{2}(1 - \alpha_0)$. Thus for $k \geq k_0$ we can write

$$W \succ \log \left(\frac{k-2}{k-1} + \exp(\widetilde{W}_0) \right) \vee 0 \geq \begin{cases} \widetilde{W}_0 & \widetilde{W}_0 \geq M \\ \alpha & M > \widetilde{W}_0 \geq -\sigma \\ 0 & -\sigma > \widetilde{W}_0 \end{cases} \quad (3.32)$$

Comparing the RHS of last equation with the definition of ν_k , it suffices show that

$$P(\widetilde{W}_0 < -\sigma) \leq \frac{1}{\log k}(1 - \kappa) - \frac{c}{\log k} \quad \text{and} \quad (3.33)$$

$$P(\widetilde{W}_0 \leq x) \leq \nu_k([0, x]) - \frac{c}{\log k} \quad \text{for all } x \geq 0 \text{ such that } \nu_k([0, x]) < 1. \quad (3.34)$$

Recall the three parts of $\tilde{\nu}_V$ in (3.27) and define $\tilde{\nu}_{W_0}^\bullet(dx) := \tilde{\nu}_V^\bullet * \tilde{\nu}_{-Z_1}(dx)$ for $\bullet \in \{1, r, \infty\}$. Figure 3.1 gives an illustration of $\tilde{\nu}_{W_0}$ and ν_k , where bars represent the discrete parts, curves represent the continuous parts and the left two dotted boxes corresponds to last two cases of (3.32). Fix $\delta = \frac{1}{2}$. To show (3.33) is to show that the weight in the first dotted box is strictly smaller than $\nu_k(\{0\}) = \kappa$. We set $\kappa = \frac{1}{2}$ such that

$$P(\text{Pois}(\kappa/2) = 0) = e^{-1/4} > \frac{3}{4} > \frac{1}{2} = \kappa.$$

Recall the definition of $C_Z = C_Z(\delta, \kappa, \alpha_0, M, \gamma)$ in (3.11). By Lemma 3.9(2), for each fixed

$\delta, \kappa, \alpha_0, M$, we can choose $\epsilon_0, \gamma_0, \beta_0$ such that for all $\epsilon < \epsilon_0, \gamma < \gamma_0, \beta_0 < \beta < 1$ and $c_0 = \frac{1}{10}$,

$$\begin{aligned} P(\widetilde{W}_0 < -\sigma) &\leq \frac{e^{\gamma+1-\beta}}{\log k} \left[P(\widetilde{Z}_1 \neq 0) + (1+\epsilon)C_Z\gamma \cdot C_{\delta,\alpha,M}^* + \epsilon \right] \\ &\leq \frac{e^{\gamma+1-\beta}}{\log k} \left[1 - e^{-\frac{1}{2}\kappa - \gamma p_{\neq}^r} + 2C_Z C_{\delta,\alpha,M}^* \gamma + \epsilon \right] \\ &= \frac{4/3}{\log k} \left[\frac{1}{4} + \epsilon + o_\gamma(1) \right] < \frac{2}{5} \frac{1}{\log k} = \left(\frac{1}{2} - c_0 \right) \frac{1}{\log k}. \end{aligned}$$

The proof of (3.34) is roughly done in three parts. We first show that the asymptotically, $\tilde{\nu}_{W_0}^r$ is smaller than ν_k^r by a multiplicative constant factor. Then we show that the underflow of $\tilde{\nu}_{W_0}^r$ below M (the vertical stripped area in Figure 3.1) can be compensated by the overflow of $\tilde{\nu}_{W_0}^1$ above M (the $q_{\widetilde{M}}$ box in Figure 3.1). Finally we make sure that the compensation is can be absorbed into the gap of $\tilde{\nu}_{W_0}^r$ and ν_k^r (the wide stripped area in Figure 3.1).

We first look at sufficiently large values of x . By Lemma 3.9(1),

$$\tilde{\nu}_{W_0}^r(dx) \leq \frac{e^{\gamma+1-\beta}}{\log k} (1 + \alpha_0)(1 + C_{\delta,M}(\gamma + \epsilon)) \exp(\kappa(e^{\alpha\delta} + 2e^{-\alpha\delta} - 3)/2) \nu_r(dx), \quad \forall x \geq M. \quad (3.35)$$

Let α_0 be a small constant such that (note that $\varphi(\frac{1}{2}) = \log 2 - o_k(1)$ and $\exp(\sqrt{2} - 3/2) \approx 0.92 < \frac{12}{13}$)

$$(1 + \alpha_0) \exp(\kappa(e^{\alpha\delta} + 2e^{-\alpha\delta} - 3)/2) = (1 + o_{\alpha_0}(1)) \exp(\sqrt{2} - 3/2) < \frac{12}{13} < 1,$$

and let $M > M(\alpha_0) \vee \frac{2}{\delta}$ such that Lemma 3.4 is satisfied. Recall the definition of constant $C_{\delta,M}$ from the constants in Lemma 3.2 and Lemma 3.9. Given our choice of $\delta, \kappa, \alpha_0, M$ so far, we can choose $\gamma_1, \epsilon_1, \beta_1$ such that for all $\gamma \leq \gamma_1, \epsilon \leq \epsilon_1, 1 - \beta < 1 - \beta_1$ and all $x \geq M$,

$$\text{RHS of (3.35)} \leq \frac{12}{13} \cdot \frac{1}{\log k} e^{1+\gamma-\beta} \nu_r(dx) \leq \frac{14}{15} \frac{1}{\log k} \nu_r(dx). \quad (3.36)$$

Next we consider the values of x near M . We first choose $\widetilde{M} = \widetilde{M}(\delta, \alpha, M) > M \vee 2\alpha$ such that

$$\frac{1}{15} \nu_r([M, \widetilde{M}]) = \frac{1}{15} \int_M^{\widetilde{M}} \frac{\gamma}{y^2} e^{\delta y} dy \geq e^{\gamma_1+1-\beta_1} (C_{\delta,\alpha,M}^* + 2e^{\gamma_1} + \epsilon_1) \gamma, \quad (3.37)$$

where $C_{\delta,\alpha,M}^*$ is the constant in Lemma 3.9(2). Let $q_{\widetilde{M}} := \frac{1}{2} P(\text{Pois}(\kappa) \cdot \alpha \in (\widetilde{M}, 2\widetilde{M}))$. $q_{\widetilde{M}}$ is strictly positive since $\widetilde{M} > 2\alpha$. By Lemma 3.9(3), we can choose σ_2, γ_2 such that for all $\sigma < \sigma_2, \gamma < \gamma_2$,

$$\tilde{\nu}_{W_0}^1([\widetilde{M}, 2\widetilde{M}]) = \tilde{\nu}_V^1 * \tilde{\nu}_{-Z_1}([\widetilde{M}, 2\widetilde{M}]) \geq q_{\widetilde{M}} > 0. \quad (3.38)$$

We further choose $\gamma_3, \epsilon_3, \beta_2$ such that for all $\gamma \leq \gamma_3, \epsilon \leq \epsilon_2 < 1, 1 - \beta \leq 1 - \beta_2$ and some $c_1 \in (0, q_{\widetilde{M}})$,

$$e^{\gamma+1-\beta} [(1 - q_{\widetilde{M}}) + \gamma C_{\delta, \alpha, M}^* + \epsilon] \leq 1 - c_1 < 1. \quad (3.39)$$

(3.36), (3.38) and (3.39) together implies for $x \leq \widetilde{M}$, (note that $\nu_k([0, M]) = 1/\log k$)

$$\begin{aligned} \tilde{\nu}_{W_0}([-\infty, x]) &:= (\tilde{\nu}_{W_0}^1 + \tilde{\nu}_{W_0}^r + \tilde{\nu}_{W_0}^\infty)([-\infty, x]) \\ &\leq \frac{e^{\gamma+1-\beta}}{\log k} ((1 - q_{\widetilde{M}}) + \gamma C_{\delta, \alpha, M}^* + \epsilon) + \frac{14}{15} \frac{1}{\log k} \nu_r([M, x \vee M]) \\ &\leq \frac{1 - c_1}{\log k} + \frac{14}{15} \frac{1}{\log k} \nu_r([M, x \vee M]) \leq \nu_k([0, x]) - \frac{c_1}{\log k}. \end{aligned}$$

Finally, for $x \geq \widetilde{M}$ such that $\nu_k([0, x]) < 1$, we can choose c_2, β_3 such that for $\gamma = (\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3)$ and $1 - \beta < 1 - \beta_3$, we have $e^{\gamma+1-\beta} + c_2 < 1 + 2\gamma e^\gamma$. Using (3.37), we have

$$\begin{aligned} \tilde{\nu}_{W_0}([-\infty, x]) &\leq \frac{e^{\gamma+1-\beta}}{\log k} (1 + \gamma C_{\delta, \alpha, M}^* + \epsilon) + \frac{14}{15} \frac{1}{\log k} \nu_r([M, \widetilde{M}]) + \frac{14}{15} \frac{1}{\log k} \nu_r((\widetilde{M}, x]) \\ &\leq \frac{1}{\log k} + \frac{1}{\log k} (e^{\gamma+1-\beta} (1 + \gamma C_{\delta, \alpha, M}^* + \epsilon) - 1 - \frac{1}{15} \nu_r([M, \widetilde{M}])) + \frac{1}{\log k} \nu_r([M, x]) \\ &\leq \frac{1 - c_2}{\log k} + \frac{1}{\log k} \nu_r([M, x]) = \nu_k([0, x]) - \frac{c_2}{k \log k}. \end{aligned}$$

Combining all pieces together, we have the desired result with $\delta, \kappa, \alpha_0, M, \gamma$ set as specified before, $\sigma = \sigma_1 \wedge \sigma_2, \epsilon = \epsilon_0 \wedge \epsilon_1 \wedge \epsilon_2$, and $\beta^0 = \beta_0 \vee \beta_1 \vee \beta_2 \vee \beta_3, c = c_0 \wedge c_1 \wedge c_2$. \square

4 Appendix

Proof of Fact 3.3. First fix $n = 2$ and $t' \geq 2M$. For each $x_1 \geq M$, either x_1 or $t' - x_1$ is larger than $t'/2$, hence

$$\int_M^{t'-M} \frac{1}{x_1^2(t - x_1)^2} dx_1 \leq \frac{2}{(t'/2)^2} \int_M^\infty \frac{1}{x_1^2} dx_1 = \frac{8}{Mt'^2}. \quad (4.1)$$

Recursively apply (4.1) with $t' = t - \sum_{i=1}^{n-j} x_i, j = 2, \dots, n-1$, we have

$$\begin{aligned} &\int_{x_i \geq M, \sum_{i=1}^{n-1} x_i \leq t-M} \frac{1}{x_1^2 \cdots x_{n-1}^2 (t - \sum_{i=1}^{n-1} x_i)^2} dx_1 \cdots dx_{n-1} \\ &= \int_{x_i \geq M, \sum_{i=1}^{n-2} x_i \leq t-2M} \frac{1}{x_1^2 \cdots x_{n-2}^2} \left(\int_M^{t - \sum_{i=1}^{n-2} x_i - M} \frac{1}{x_{n-1}^2 (t - \sum_{i=1}^{n-1} x_i)^2} dx_{n-1} \right) dx_1 \cdots dx_{n-2} \\ &\leq \frac{8}{M} \int_{x_i \geq M, \sum_{i=1}^{n-2} x_i \leq t-M} \frac{1}{x_1^2 \cdots x_{n-2}^2 (t - \sum_{i=1}^{n-2} x_i)^2} dx_1 \cdots dx_{n-2} \leq \cdots \leq \left(\frac{8}{M}\right)^n \frac{1}{t^2}. \end{aligned}$$

□

Proof of Fact 3.7. Let $s_k = (\frac{\gamma^\delta}{\log k (\log \log k)^2})^{1/\delta}$, it is easy to check that

$$\frac{\gamma}{\delta} s_k^{-\delta} \log^{-2} s_k = (1 + o_k(1)) \log k.$$

For any $\epsilon > 0$, let c be large enough such that $(1 - \epsilon)^\delta - 2c^{-\delta} > 1$. It follows that

$$\begin{aligned} \int_{(1-\epsilon)s_k}^{\infty} k \log k \cdot \mu_U(dx) &= \int_{(1-\epsilon)s_k}^{\psi(M)} \frac{\gamma}{(\log x)^2} x^{-(1+\delta)} dx \geq \frac{\gamma}{\log^2(1-\epsilon)s_k} \int_{(1-\epsilon)s_k}^{cs_k} x^{-(1+\delta)} dx \\ &= \frac{\gamma}{\delta \log^2(1-\epsilon)s_k} s_k^{-\delta} ((1-\epsilon)^{-\delta} - c^{-\delta}) > (1 + c^{-\delta} + o_k(1)) \log k. \end{aligned}$$

Therefore $t_k > (1 - \epsilon)s_k$ for $k \geq k_0$. In the other direction, let $s'_k = (c' \log k)^{-1/\delta}$ for some large constant $c' > 0$, $\log(s'_k) = (1 + o_k(1)) \frac{1}{\delta} \log \log k = (1 + o_k(1)) \log s_k$, we have

$$\begin{aligned} \int_{(1+\epsilon)s_k}^{\infty} k \log k \cdot \mu_U(dx) &= \int_{(1+\epsilon)s_k}^{\psi(M)} \frac{\gamma}{(\log x)^2} \frac{1}{x^{1+\delta}} dx \\ &\leq \frac{\gamma}{\log^2 \psi(M)} \int_{s'_k}^{\infty} x^{-(1+\delta)} dx + \frac{\gamma}{\log^2(s'_k)} \int_{(1+\epsilon)s_k}^{\infty} x^{-(1+\delta)} dx \\ &\leq \frac{\gamma}{\delta \log^2 \psi(M)} c'^{-\delta} \log k + (1 + o_k(1))(1 + \epsilon)^{-\delta} \log k. \end{aligned}$$

Let c' be large enough such that $\frac{\gamma}{\delta \log^2 \psi(M)} c'^{-\delta} + (1 + \epsilon)^{-\delta} < 1 - c'^{-1} < 1$, we have for $k \geq k_0$ that $t_k < (1 + \epsilon)s_k$. This completes the Part 1. Part 2 can be derived similarly. □

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